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Symmetry Plane and Relativity

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Abstract: Euclidean geometry, inherited from ancient Greece, was modeled on axiomatic methods in modern science. Hilbert's "Foundations of Geometry" supplemented the lacking axioms, and seemed to have reached the stage of completion of plane geometry, but still questions remain why there is no definition of a plane nor a line. Looking back on the history of special relativity, Lorentz and Poincaré were on their way to finding a theory to prove the results of Michelson-Morley experiment. Meanwhile, Einstein published the theory of relativity based on the two principles. At a glance, all things have been done, but this is not enough. Following the Poincaré conjecture [1,2] and digging into why the relativity principle holds, we arrive at a deeper symmetry of spacetime. A paragraph of Hannyashin Sutra "空即是色 Kuu soku ze shiki" is interpreted as "emptiness contains the form of cosmos". From the viewpoint of spacetime substantivalism, empty space is a treasure trove in which to discover the hidden rules of the cosmos. Reading the book of Nature written in mathematics, we observe that fundamental symmetry is a plane with indistinguishable back and front surfaces in which the basic laws must be subject to this symmetry.

Keywords: symmetry plane; symmetric plane; invariant function; inner product; Minkowski spacetime; relativity principle; arrow of time

1. Introduction

1.1. My basic questions from school days

- For plane geometry

Why does the axiom system not depart from the properties of a plane itself [3,4]? A plane is two-dimensional linear space, with back and front symmetry. It is concerned that ancient field of view of a plane that was drawing figures on the ground still continues.

- For linear algebra

Why is the inner product not deduced from a Euclidean plane, but defined on a vector space? A Euclidean plane belongs to nature. For mathematics to make sense, it is essential to give the meaning of the inner product not only from the form itself but also from the internal harmony within it.

- For the theory of special relativity

It seems that the two principles are not independent. What can be considered are (1) one is contained in the other, or (2) there is a deeper principle that applies to both parties. The nature of spacetime, especially unidirection of time and symmetry of plane should be involved in the theory.

1.2. How to prove that "a Euclidean plane is inversion invariant for any line on itself" ?

Proof【Put an origin on any point in a Euclidean plane. The rotation matrix A and the reflection matrix B are established on a Euclidean plane as $A=\begin{pmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{pmatrix}$ and $B=\begin{pmatrix}\cos\theta & -\sin\theta\\ -\sin\theta & -\cos\theta\end{pmatrix}$. Note that $B=B^{-1} \Leftrightarrow B^2=E$, $A=BM$, $M=\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$, and $E=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$. In the Figure 1., let there be a coordinate axis x_2-y_2 by matrix B on the back side and x-y on the front side, and x_A-y_A by matrix A on the front side of a plane. From the reflection matrix B, their relations are

y_2 axis : $x_2=\cos\theta\cdot x-\sin\theta\cdot y=0 \Leftrightarrow y=\cot\theta\cdot x=x/\sqrt{3}$, where $\theta=\pi/3$ for example,

x_2 axis : $y_2 = -\sin\theta \cdot x - \cos\theta \cdot y = 0 \Leftrightarrow y = -\tan\theta \cdot x = -\sqrt{3}x$.

The matrix B transforms a point $\boldsymbol{p}$ on the front side to the corresponding rear point $\boldsymbol{q}$ on the back side as $\boldsymbol{q} = B\boldsymbol{p}$, and also to the reflection point $\boldsymbol{q}$ on the front side as $\boldsymbol{q} = B\boldsymbol{p}$.

The eigen values, eigen lines through the origin, and eigen plane of matrix B are as follows:

- Eigen values are $\lambda = \pm 1$, as trace $B = 0$ and det $B = -1$.
- For $\lambda = 1 \Leftrightarrow$ a fixed-point equation $B\boldsymbol{p} = \boldsymbol{p}$, this eigen line is named a fold line f:

$$\cos\theta \cdot x - \sin\theta \cdot y = x \Leftrightarrow y = \frac{\cos\theta - 1}{\sin\theta} \cdot x = -\tan\frac{\theta}{2} \cdot x = -x/\sqrt{3}. \quad (1)$$

- For $\lambda = -1 \Leftrightarrow$ an inversion equation $B\boldsymbol{p} = -\boldsymbol{p}$, this eigen line is named an isotropic line g :

$$\cos\theta \cdot x - \sin\theta \cdot y = -x \Leftrightarrow y = \frac{\cos\theta + 1}{\sin\theta} x = \cot\frac{\theta}{2} \cdot x = \sqrt{3}x.$$

- Eigen plane is made of eigen lines f and g, and it is semi-isotropic, since the line segment ($\boldsymbol{p} - B\boldsymbol{p}$) is parallel to the isotropic line g, and its middle point is in the fold line f.

The point $\boldsymbol{p}$ on the front side is transformed by B as $\boldsymbol{p}$(front)$\rightarrow B\boldsymbol{p}$(front)$\rightarrow B^2\boldsymbol{p} = \boldsymbol{p}$(back).

The point $\boldsymbol{p}$ on back side is transformed by B as $\boldsymbol{p}$(back)$\rightarrow B\boldsymbol{p}$(back)$\rightarrow B^2\boldsymbol{p} = \boldsymbol{p}$(front).

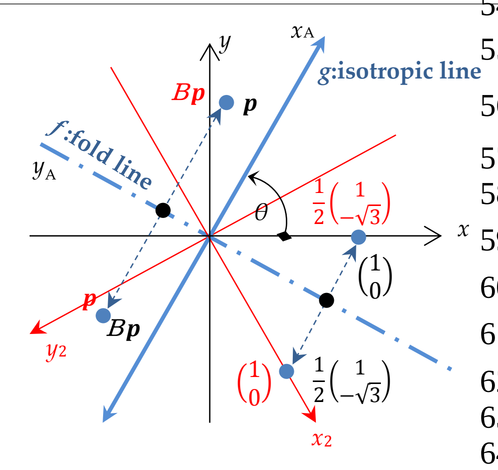

Figure 1. Euclidean plane

Example: $\theta = \pi/3$, $\tan\theta = \sqrt{3}$, $\tan\frac{\theta}{2} = 1/\sqrt{3}$

$$B = \begin{pmatrix} \cos\theta & -\sin\theta \\ -\sin\theta & -\cos\theta \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & -\sqrt{3} \\ -\sqrt{3} & -1 \end{pmatrix}.$$

front side		**back side**
$\boldsymbol{p} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$	$\rightleftarrows$	$B\boldsymbol{p} = \frac{1}{2}\begin{pmatrix} 1 \\ -\sqrt{3} \end{pmatrix}$
$\uparrow \downarrow$		$\uparrow \downarrow$ *
$B\boldsymbol{p} = \frac{1}{2}\begin{pmatrix} 1 \\ -\sqrt{3} \end{pmatrix}$	$\rightleftarrows$	$\boldsymbol{p} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = B^2\boldsymbol{p}$

$\boldsymbol{p} \rightarrow B\boldsymbol{p} \rightarrow B^2\boldsymbol{p} = \boldsymbol{p}$

$B^2\boldsymbol{p} = \boldsymbol{p} \leftarrow B\boldsymbol{p} \leftarrow \boldsymbol{p}$

*Note that front→front means figure transformation and front⇄back means coordinate transformation.

Therefore, the point $\boldsymbol{p}$ on the front side is equivalent to the point $\boldsymbol{p}$ on the back side, so the eigen plane is symmetric for the fold line f as an axis of reflection. Since the direction of a hold line f can be in all directions as per Equation (1), then "a Euclidean plane is inversion invariant for any line on itself".□】

The inverse proposition that "If a plane is symmetric, then we have a Euclidean plane" is partially true, as given in the next section.

1.3. What does the symmetry of a plane deduce?

Put right-hand oblique coordinate systems on both face sides of a plane, and make them coincide with their origins. We define the 2×2 rear surface coordinate transformation matrix B as an inside out transformation, then det $B < 0$. Because it is not possible to distinguish which side of a plane is the back or the front, the symmetry plane equation is $B = B^{-1} \Leftrightarrow B^2 = E$. We obtain an oblique reflection transformation matrix B with two degrees of freedom:

$$B = \begin{pmatrix} a & -b \\ c & -a \end{pmatrix} = \begin{pmatrix} a & -b \\ kb & -a \end{pmatrix}, \text{ where } \det B = -1,\ k = c/b.$$

We derive the matrix $A = BM = \begin{pmatrix} a & -b \\ c & -a \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} a & b \\ c & a \end{pmatrix} = \begin{pmatrix} a & b \\ kb & a \end{pmatrix}$, where det $A = 1$. It is known that if $k = -1$, then A is a rotation matrix and B is a reflection matrix. If $k < 0$, $k = 0$, and $k > 0$ in order, then the matrix A is referred to as elliptic transformation, Galilean transformation, and Lorentz transformation respectively. When we fix coefficient k, then matrices A and B create an isometric transformation group (see Equation (38) and (39)).

Thus, the symmetry of a plane gives rise not only to Euclidean geometry when $k=-1$, but also to the principle of relativity when $k \geq 0$.

2. Terms, definitions, axioms, and mathematical preparations

Terms:

- An *oblique reflection plane* has a fold line and isotropic lines. A point $\boldsymbol{p}$ is transformed to a point $\boldsymbol{Bp}$ in the same isotropic line by an *oblique reflection transformation* $\boldsymbol{B}$, and their middle point is in the fold line (see Figure 2.).

Definitions:

- *Spacetime* is a four-dimensional unified entity of space and time without considering all of the matter from the universe.
- *Space* is continuous, infinite, homogeneous, three-dimensional, and isotropic.
- *Time* is continuous, infinite, homogeneous, one-dimensional and unidirectional, and irreversible.
- A *line* in spacetime is one-dimensional, and it is inversion invariant for any point on itself.
- A *plane* in spacetime is two-dimensional, and it is inversion invariant for any axis of a fold line passing through any two points on itself.
- An *asymmetric plane* is a plane with distinguishable back and front surfaces.
- A *symmetry plane* or a *symmetric plane* is a plane with indistinguishable back and front surfaces.
- A line of a coordinate axis on a plane in spacetime is one of two types, namely, a *space line* is isotropic, or a *time line* is unidirectional.
- A plane in spacetime is one of two types, namely, *space* × *space* type or *space* × *time* type.
- A space × space type plane is completely isotropic, since the space line is isotropic. This type of plane exists as a subspace of a three-dimensional space in an inertial system.
- A space × time type plane is semi-isotropic, since the time line is of unidirectional and the space line is isotropic. This type of plane exists when we think of one-dimensional space in which all inertial systems move on one line and inertial coordinate systems with a space-axis and a time-axis coexist in one common space × time plane. Note that "*spacetime*" means four-dimensional space of space- time, and "*space* × *time*" means two-dimensional plane.
- *Invariant function*: Let $\boldsymbol{A}=\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be a matrix, $\boldsymbol{p}=\begin{pmatrix} x \\ y \end{pmatrix}$ be a point, and f be a function of $\boldsymbol{p}$. If $\mathrm{f}(\boldsymbol{Ap}) = \mathrm{f}(\boldsymbol{p})$, then this function $\mathrm{f}(\boldsymbol{p})$ is called an invariant function of $\boldsymbol{A}$.

Axioms:

- *Inertial system axiom*: There are an infinite number of empty inertial coordinate systems (or inertial systems) in empty spacetime. Each has its own four-dimensional spacetime and keeps its uniform motion on a straight line.
- *Symmetry plane axiom*: It is not possible to distinguish which side of a plane in spacetime is the back or the front.

Mathematical preparations:

- An *invariant line* $\mathrm{f}(\boldsymbol{p})$ of a 2×2 matrix $\boldsymbol{B}=\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is the solution to a first order invariant function equation in the form of $\mathrm{f}(\boldsymbol{Bp}) \equiv \mathrm{f}(\boldsymbol{p}) = ux+vy$

  $\Leftrightarrow u(ax+by)+v(cx+dy) = ux+vy \Leftrightarrow [(a-1)u+cv]\,x+[bu+(d-1)v]\,y = 0$

  $\Leftrightarrow x,y$ are arbitrary, and in order to have a non-self-explanatory solution,

$$\text{the determinant} = (a-1)(d-1)-bc = 0. \quad (2)$$

When the matrix $\boldsymbol{B}$ has an eigen value $\lambda=1$, we obtain an invariant line $\mathrm{f}(\boldsymbol{p})$ by substituting $u=c$, $v=-(a-1)$ on $\mathrm{f}(\boldsymbol{p}) = ux+vy$. Thus, an invariant line is

$$\mathrm{f}(\boldsymbol{Bp}) \equiv \mathrm{f}(\boldsymbol{p}) = cx-(a-1)y. \quad (3)$$

- *Quadratic invariant function* $\phi(\boldsymbol{p})$ of a 2×2 matrix $\boldsymbol{A}=\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is the solution of a second order invariant function equation in the form of

$$\phi(Ap) \equiv \phi(p) = ux^2+vy^2+wxy \Leftrightarrow$$

$$\phi(Ap) = \phi(ax+by,\ cx+dy) = u(ax+by)^2+v(cx+dy)^2+w(ax+by)(cx+dy) = ux^2+vy^2+wxy \quad (4)$$

$$\Leftrightarrow [(a^2-1)u+c^2v+acw]\,x^2+[b^2u+(d^2-1)v+bdw]\,y^2+[2ab+2cdv+(ad+bc-1)w]\,xy = 0.$$

Since x^2, y^2, and xy are arbitrary, each coefficient must be equal to 0, and we obtain a simultaneous equation of u, v, and w,

$$\begin{pmatrix} a^2-1 & c^2 & ac \\ b^2 & d^2-1 & bd \\ 2ab & 2cd & ad+bc-1 \end{pmatrix}\begin{pmatrix} u \\ v \\ w \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}. \quad (5)$$

In order to have a non-self-explanatory solution, the determinant of the matrix in Equation (5) must be equal to 0. Thus,

$$\text{the determinant} = (ad-bc-1)[(ad-bc+1)^2-(a+d)^2] = 0$$
$$\Leftrightarrow ad-bc = 1 \text{ or } ad-bc = -1 \text{ and } a+d = 0. \quad (6)$$

Here, by using the solution of $u=-c$, $v=b$, $w=a-d$, we obtain an *identity*:

$$\phi(Ap) \equiv \det A \cdot \phi(p), \text{ where } A \text{ is a matrix } A= \begin{pmatrix} a & b \\ c & d \end{pmatrix}, p \text{ is a point } p=\begin{pmatrix} x \\ y \end{pmatrix},$$
$$\text{and } \phi(p) \text{ is a quadratic function given by } \phi(p) = -cx^2+by^2+(a-d)xy. \quad (7)$$

(1) From Equations (6) and (7), if $\det A=1$, then

$$\phi(Ap) = \phi(p) = -cx^2+by^2+(a-d)xy. \quad (8)$$

In this case, $\phi(p)$ is the second order invariant function of the matrix A. $\phi(p)$ is allowed to be multiplied by a scale factor, such that

$$\text{if } \Phi(p) = r\,\phi(p), \text{ then } \underline{\Phi(Ap)} = r\,\phi(Ap) = r \det A\,\phi(p) = r\,\phi(p) = \underline{\Phi(p)}. \quad (9)$$

(2) Moreover, if $\det A \neq 1$, then $\phi(p)$ is called a *relative invariant function* of matrix A.
(3) We also obtain another invariant function from Equation (6). If $\det A=-1$ and $\mathrm{trace} A=a+d=0 \Leftrightarrow$ if eigen values of A are $\lambda=\pm 1$, then change the notation of matrix A into B for convenience, and the invariant function has the same part of Equation (8) with the cross product xy eliminated, such that

$$\text{when } B= \begin{pmatrix} -a & -b \\ c & a \end{pmatrix}, \text{ and } \det B = -1, \text{ then } \phi(Bp) = \phi(p) = -cx^2+by^2. \quad (10)$$

- A special linear transformation matrix S has *commutative coefficients k,h* and is disassembled as

$$S=\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} m+hb & b \\ kb & m-hb \end{pmatrix},$$
$$\text{where } \det S = m^2-\Delta b^2 = 1,\ \Delta= h^2+k,\ m = (a+d)/2,\ k = c/b,\ 2h = (a-d)/b, \text{ and } b \neq 0. \quad (11)$$

$$\text{When } S_1=\begin{pmatrix} m_1+hb_1 & b_1 \\ kb_1 & m_1-hb_1 \end{pmatrix}, \text{ and } S_2=\begin{pmatrix} m_2+hb_2 & b_2 \\ kb_2 & m_2-hb_2 \end{pmatrix}, \text{ then matrices } S_1, S_2,$$
$$\text{and their products } S_1S_2 \text{ have common coefficients } k, h, \text{ and } S_1S_2=S_2S_1 \text{ holds.} \quad (12)$$

From Equations (8) and (9), we have that matrix S has a normalized invariant function,

$$\phi(Sp) = \phi(p) = -kx^2+y^2+2hxy. \quad (13)$$

The matrix S and the invariant function $\phi(p)$ can be classified into three types based on the sign of the discriminant $\Delta=h^2+k$.
If $\Delta< 0$, then they are of an elliptic type.
If $\Delta> 0$, then they are of a hyperbolic type.
If $\Delta= 0$, then they are of a linear type.
Thus, we define the polar form of 2×2 special matrix $S= \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} m+hb & b \\ kb & m-hb \end{pmatrix}$, where

$\det \boldsymbol{S}=m^2-\Delta b^2=1$, using *argument* θ, and commutative coefficient k, h as follows:

$$\text{For } \Delta<0,\quad \boldsymbol{S}=\boldsymbol{S}(\theta,k,h)=\begin{pmatrix}\cos\theta+\frac{h}{\sqrt{-\Delta}}\sin\theta & \frac{1}{\sqrt{-\Delta}}\sin\theta\\ \frac{k}{\sqrt{-\Delta}}\sin\theta & \cos\theta-\frac{h}{\sqrt{-\Delta}}\sin\theta\end{pmatrix}, \text{ elliptic type.} \tag{14}$$

$$\text{For } \Delta>0,\quad \boldsymbol{S}=\boldsymbol{S}(\theta,k,h)=\begin{pmatrix}\cosh\theta+\frac{h}{\sqrt{\Delta}}\sinh\theta & \frac{1}{\sqrt{\Delta}}\sinh\theta\\ \frac{k}{\sqrt{\Delta}}\sinh\theta & \cosh\theta-\frac{h}{\sqrt{\Delta}}\sinh\theta\end{pmatrix}, \text{ hyperbolic type.} \tag{15}$$

$$\text{For } \Delta=0,\quad \boldsymbol{S}=\boldsymbol{S}(b,h)=\begin{pmatrix}m+hb & b\\ -h^2b & m-hb\end{pmatrix}, \text{ where } m=\pm 1, \text{ linear type.} \tag{16}$$

Any 2×2 non-diagonal regular matrix $\boldsymbol{F}$ is represented by the *polar form* of

$$\boldsymbol{F}=(\det \boldsymbol{F})^{1/2}\,\boldsymbol{S}(\theta,k,h) \text{ or } \boldsymbol{F}=(\det \boldsymbol{F})^{1/2}\,\boldsymbol{S}(b,h). \tag{17}$$

However, when $\det \boldsymbol{F}<0$, the matrix $\boldsymbol{F}$ represents an inside out transformation, and then the orbit of the invariant function as shown Equation (13) branches off to a conjugate curve of $\phi(\boldsymbol{p})$, and complex number of argument θ appears.

We obtain the *addition theorem* of argument θ or b from Equations (14)−(16) as follows:

$$\begin{gathered}\boldsymbol{S}(\theta_1,k,h)\boldsymbol{S}(\theta_2,k,h)=\boldsymbol{S}(\theta_1+\theta_2,k,h),\ \boldsymbol{S}(\theta,k,h)^{n}=\boldsymbol{S}(n\theta,k,h),\ \boldsymbol{S}(\theta,k,h)^{-1}=\boldsymbol{S}(-\theta,k,h),\\ \boldsymbol{S}(b_1,h)\boldsymbol{S}(b_2,h)=\boldsymbol{S}(b_1+b_2,h),\ \boldsymbol{S}(b,h)^{n}=\boldsymbol{S}(nb,h),\ \boldsymbol{S}(b,h)^{-1}=\boldsymbol{S}(-b,h).\end{gathered} \tag{18}$$

- The *norm* $\|\boldsymbol{p}\|$ of a vector $\boldsymbol{p}$ is defined by the invariant function $\phi(\boldsymbol{p})$ such that

$$\|\boldsymbol{p}\|^2=\phi(\boldsymbol{p})=-kx^2+y^2+2hxy, \text{ and the norm } \|\boldsymbol{p}\|=\phi(\boldsymbol{p})^{1/2}. \tag{19}$$

- The *inner product* of vector $\boldsymbol{p}$ and $\boldsymbol{q}$ is defined by the invariant function $\phi(\boldsymbol{p})$ as follows:

$$\boldsymbol{p}=(x_1,y_1),\ \boldsymbol{q}=(x_2,y_2)=\boldsymbol{F}\boldsymbol{p}=(\det \boldsymbol{F})^{1/2}\boldsymbol{S}(\theta,k,h)\,\boldsymbol{p}, \tag{20}$$

$$\begin{aligned}&\|\boldsymbol{p}\|^2=\phi(\boldsymbol{p})=-kx_1^2+y_1^2+2hx_1y_1,\ \|\boldsymbol{q}\|^2=\phi(\boldsymbol{q})=-kx_2^2+y_2^2+2hx_2y_2,\\ &\|\boldsymbol{p}+\boldsymbol{q}\|^2=\phi(\boldsymbol{p}+\boldsymbol{q})=-k(x_1+x_2)^2+(y_1+y_2)^2+2h(x_1+x_2)(y_1+y_2),\\ &=\|\boldsymbol{p}\|^2+\|\boldsymbol{q}\|^2+2(\underline{-kx_1x_2+y_1y_2+h(x_1y_2+x_2y_1)}).\end{aligned} \tag{21}$$

Thus, we induce the inner product and the *cosine theorems* from Equations (14), (15), and (20),

$$\begin{aligned}(\boldsymbol{p},\boldsymbol{q})&=\boldsymbol{p}\cdot\boldsymbol{q}=-kx_1x_2+y_1y_2+h(x_1y_2+x_2y_1)=\boldsymbol{p}\cdot(\det \boldsymbol{F})^{1/2}\,\boldsymbol{S}(\theta,k,h)\boldsymbol{p}\\ &=\tfrac{1}{2}(\|\boldsymbol{p}+\boldsymbol{q}\|^2-\|\boldsymbol{p}\|^2-\|\boldsymbol{q}\|^2)=\tfrac{1}{2}(\phi(\boldsymbol{p}+\boldsymbol{q})-\phi(\boldsymbol{p})-\phi(\boldsymbol{q}))\\ &=(\det \boldsymbol{F})^{1/2}\phi(\boldsymbol{p})\cos\theta=\|\boldsymbol{p}\|\,\|\boldsymbol{q}\|\cos\theta, \text{ when } \boldsymbol{S} \text{ is an elliptic type,}\\ &=(\det \boldsymbol{F})^{1/2}\phi(\boldsymbol{p})\cosh\theta=\|\boldsymbol{p}\|\,\|\boldsymbol{q}\|\cosh\theta, \text{ when } \boldsymbol{S} \text{ is a hyperbolic type.}\end{aligned} \tag{22}$$

- Furthermore, when $d=a \Leftrightarrow h=0$ on a special linear transformation $\boldsymbol{S}$, we define a commutative special *isodiagonal* transformation $\boldsymbol{A}$ and invariant function $\phi(\boldsymbol{p})$ given by

$$\begin{gathered}\boldsymbol{A}=\begin{pmatrix}a & b\\ c & a\end{pmatrix}=\begin{pmatrix}a & b\\ kb & a\end{pmatrix}, \det \boldsymbol{A}=a^2-kb^2=1,\ k=c/b, \text{ and}\\ \phi(\boldsymbol{A}\boldsymbol{p})=\phi(\boldsymbol{p})=-kx^2+y^2, \text{ where } k \text{ is a commutative coefficient.}\end{gathered} \tag{23}$$

In this case, we define the norm $\|\boldsymbol{p}\|$ and the inner product $(\boldsymbol{p},\boldsymbol{q})$ as follows:

$$\|\boldsymbol{p}\|^2=\phi(\boldsymbol{p})=-kx^2+y^2,\ \|\boldsymbol{p}\|=\phi(\boldsymbol{p})^{1/2}, \tag{24}$$

$$(\boldsymbol{p},\boldsymbol{q})=\boldsymbol{p}\cdot\boldsymbol{q}=-kx_1x_2+y_1y_2. \tag{25}$$

If $(\boldsymbol{p},\boldsymbol{q}) = 0 \Leftrightarrow (y_1/x_1)(y_2/x_2) = k$, then the vectors of $\boldsymbol{p}$ and $\boldsymbol{q}$ are defined as orthogonal. (26)

We obtain the polar form of $\boldsymbol{A}$ from Equations (14)−(16).

$$\text{When } k<0, \text{ then } \boldsymbol{A}=\begin{pmatrix} a & b \\ kb & a \end{pmatrix}=\begin{pmatrix} \cos\theta & \sin\theta/\sqrt{-k} \\ -\sqrt{-k}\sin\theta & \cos\theta \end{pmatrix}, \theta \text{ is an elliptic angle.} \quad (27)$$

When $k=-1$, then this type of matrix $\boldsymbol{A}$ is called a rotation transformation.

$$\text{When } k>0, \text{ then } \boldsymbol{A}=\begin{pmatrix} a & b \\ kb & a \end{pmatrix}=\begin{pmatrix} \cosh\theta & \sinh\theta/\sqrt{k} \\ \sqrt{k}\sinh\theta & \cosh\theta \end{pmatrix}, \theta \text{ is a hyperbolic angle.} \quad (28)$$

This type of matrix $\boldsymbol{A}$ is called a Lorentz transformation.

$$\text{When } k=0, \text{ then } \boldsymbol{A}=\begin{pmatrix} a & b \\ kb & a \end{pmatrix}=\begin{pmatrix} a & b \\ 0 & a \end{pmatrix}, a=\pm 1. \quad (29)$$

This type of matrix $\boldsymbol{A}$ is called a Galilean transformation.

3. Geometric structure of a line

Put two number lines 1 and 2 on one line, and make them coincide with their origins. The relation between their x-coordinates of x_1 and x_2 is $x_2=rx_1 \Leftrightarrow x_1=r^{-1}x_2$, where r is a proportional constant. For two equivalent number lines, we must have

$$r=r^{-1} \Leftrightarrow r^2=1 \Leftrightarrow r=\pm 1.$$

- When $r=-1$, then the two number lines are inversion of each other, and this type of line is isotropic. A space line fits into this category.
- When $r=1$, then the two number lines coincide with each other, and this type of line is one way. A time line fits into this category.
- When $r\neq\pm 1$, then the two number lines are similar.

4. Geometric structure of a plane

Theorem: A symmetry plane is a linear space.
Brief proof【 From the definition of the inversion invariance of a line, a line is a linear space. Also from the definition of a plane, we obtain at least two lines that exist in a plane. Then, from the inversion invariance of a plane, we observe that these lines make a plane linear. □】

Put right-hand oblique coordinate systems on both face sides of a plane, and make them coincide with their origins. We define the 2×2 rear surface coordinate transformation matrix $\boldsymbol{B}$ as an inside out transformation, then $\det \boldsymbol{B}<0$.

If $\boldsymbol{B}\neq\boldsymbol{B}^{-1}$, then this plane is not symmetric, while if $\boldsymbol{B}=\boldsymbol{B}^{-1}$, then this plane is symmetric. Therefore, the symmetry plane equation is

$$\boldsymbol{B}=\boldsymbol{B}^{-1} \Leftrightarrow \boldsymbol{B}^2=\boldsymbol{E}, \text{ where } \det\boldsymbol{B}<0. \quad (30)$$

We obtain an oblique reflection transformation matrix $\boldsymbol{B}$ with two degrees of freedom:

$$\boldsymbol{B}=\pm\begin{pmatrix} -a & -b \\ c & a \end{pmatrix}=\pm\begin{pmatrix} -a & -b \\ kb & a \end{pmatrix}, \det\boldsymbol{B}=-a^2+bc=-1, k=c/b, \text{ eigen values } \lambda=\pm 1. \quad (31)$$

The matrix $\boldsymbol{B}$ has the following properties (we shall treat the negative solution $-\boldsymbol{B}$ later).

- As the matrix $\boldsymbol{B}$ has an eigen value of 1, then it has an invariant line $f(\boldsymbol{p})$ like as Equation (3).

$$f(\boldsymbol{B}\boldsymbol{p}) \equiv f(\boldsymbol{p}) = cx+(a+1)y. \quad (32)$$

- For $\lambda=1 \Leftrightarrow$ a fixed-point equation $\boldsymbol{B}\boldsymbol{p}=\boldsymbol{p}$, this eigen line is called a fold line f:

$$cx+(a-1)y=0. \quad (33)$$

- For $\lambda=-1 \Leftrightarrow$ an inversion equation $\boldsymbol{B}\boldsymbol{p}=-\boldsymbol{p}$, this eigen line is called an isotropic line g:

$$cx+(a+1)y=0. \quad (34)$$

This line g is isotropic regarding its origin, and is parallel to an invariant line f($\boldsymbol{p}$) given as per Equation (32).

When a point $\boldsymbol{p}$ is in an invariant line f($\boldsymbol{p}$), and the point $\boldsymbol{r}$ is the intersection point of a fold line f and an invariant line f($\boldsymbol{p}$), then in the fold line f, $B\boldsymbol{r}$=$\boldsymbol{r}$, and in the invariant line, f($B\boldsymbol{p}$)=f($\boldsymbol{p}$)=f($\boldsymbol{r}$). Obtained by translating the vector ($\boldsymbol{p}-\boldsymbol{r}$) onto the isotropic line g,

$$B(\boldsymbol{p}-\boldsymbol{r}) = -(\boldsymbol{p}-\boldsymbol{r}) \Leftrightarrow B\boldsymbol{p}-\boldsymbol{r} = -\boldsymbol{p}+\boldsymbol{r} \Leftrightarrow B\boldsymbol{p}+\boldsymbol{p} = 2\boldsymbol{r}. \quad (35)$$

Since the fixed-point $\boldsymbol{r}$ is the middle point of the point $\boldsymbol{p}$ and $B\boldsymbol{p}$, and each invariant line f($\boldsymbol{p}$) is parallel to the isotropic line g, then the invariant lines f($\boldsymbol{p}$) are isotropic. The inner product of f and g is $\frac{-c}{a-1}\ \frac{-c}{a+1} = \frac{c^2}{a^2-1} = \frac{c^2}{bc} = \frac{c}{b} = k$. From Equation(26), these two lines of f and g are orthogonal, but commonly seem not perpendicular. The eigen plane with eigen lines of f and g is called an oblique reflection plane, and it is semi-isotropic. The point $B\boldsymbol{p}$ on the back side is hidden behind the point $\boldsymbol{p}$ on the front side.

Figure 2. shows the case of $B = \frac{1}{3}\begin{pmatrix} -5 & 4 \\ -4 & 5 \end{pmatrix}$, k=1.

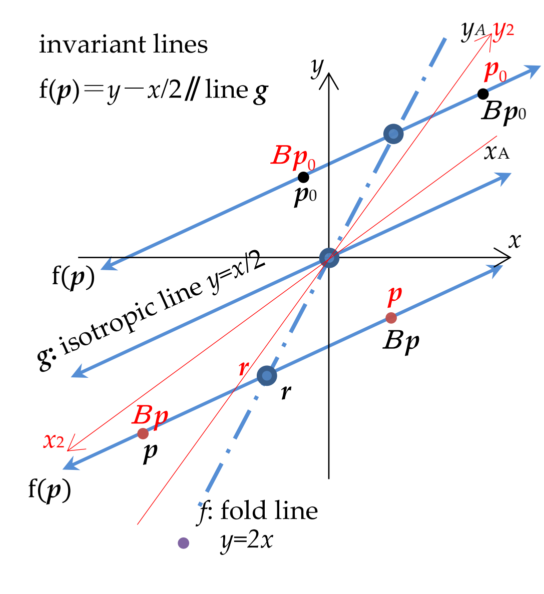

Figure 2. Oblique reflection plane

- Meanwhile, when $B \neq B^{-1}$, then we have another geometry on an asymmetric plane.
- We derive a special *isodiagonal* transformation matrix A from the oblique reflection transformation matrix B and the reflection matrix M such that

$$A=MB = M\begin{pmatrix} -a & -b \\ c & a \end{pmatrix} = \begin{pmatrix} a & b \\ c & a \end{pmatrix} = \begin{pmatrix} a & b \\ kb & a \end{pmatrix}, \text{ where } M = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \det A = 1,\ k = c/b. \quad (36)$$

The matrix A is a coordinate transformation between the right-hand systems in which the left-hand x_2-y_2 system on the back side is reflected in the right-hand x_A-y_A system on the front side by M. Note that $A=\pm BM$ or $A=\pm MB$ is equivalent as to left to right-hand system inversion.

Since B=MA, $\phi(M\boldsymbol{p}) = \phi(\boldsymbol{p})$, and $\phi(A\boldsymbol{p}) = \phi(\boldsymbol{p}) = -kx^2+y^2$ as per Equation (23), then the matrix B has the same invariant function $\phi(\boldsymbol{p})$ of A which is already implied in Equation (10),

$$\phi(B\boldsymbol{p}) = \phi(M(A\boldsymbol{p})) = \phi(A\boldsymbol{p}) = \phi(\boldsymbol{p}) = -kx^2+y^2,\ k = c/b. \quad (37)$$

When the commutative coefficient k is fixed, then we observe that any combination of matrices B and A has the common invariant function $\phi(\boldsymbol{p})$, and their joint operation is closed in the orbit of $\phi(\boldsymbol{p})$ as

$$\begin{gathered} \phi(BA^2 \ldots B^{-1}A^{-1}\boldsymbol{p}) = \phi(A^2 \ldots B^{-1}A^{-1}\boldsymbol{p}) = \phi(A \ldots B^{-1}A^{-1}\boldsymbol{p}) = \phi(B^{-1}A^{-1}\boldsymbol{p}) \\ = \phi(BA^{-1}\boldsymbol{p}) = \phi(A^{-1}\boldsymbol{p}) = \phi(A\boldsymbol{p}) = \phi(\boldsymbol{p}) = \phi(E\boldsymbol{p}) = -kx^2+y^2, \text{ where } B^{-1} = B. \end{gathered} \quad (38)$$

Thus, we conclude that any combination of B and A creates an *isometric transformation group* on the orbit of invariant function $\phi(\boldsymbol{p})$ with the metric

$$\| B\boldsymbol{p} \|^2 = \| A\boldsymbol{p} \|^2 = \| \boldsymbol{p} \|^2 = \phi(B\boldsymbol{p}) = \phi(A\boldsymbol{p}) = \phi(\boldsymbol{p}) = -kx^2+y^2. \quad (39)$$

The oblique reflection matrix B transforms a point $\boldsymbol{p}$ on the front side to the corresponding rear point $\boldsymbol{q}$ on the back side as

$$\boldsymbol{q}_1 = B\boldsymbol{p}_1,\ \boldsymbol{q}_2 = B\boldsymbol{p}_2. \quad (40)$$

However, on the front side, a figure transformation matrix X transforms a point from $\boldsymbol{p}_1$ to $\boldsymbol{p}_2$. as

$$\boldsymbol{p}_2 = X\boldsymbol{p}_1,\ \det X > 0. \quad (41)$$

Also on the back side, Y transforms a point from q_1 to q_2 as

$$q_2 = Yq_1,\ \det Y > 0. \tag{42}$$

Consequently, from these four equations, we obtain the relation,

$$q_2 = Yq_1 = \underline{YBp_1} = Bp_2 = \underline{BXp_1}. \tag{43}$$

Since the point p_1 is arbitrary and $B = B^{-1}$, then we obtain

$$YB = BX \Leftrightarrow Y = BXB \Leftrightarrow BY = XB, \quad \text{where } \det Y = \det X > 0,\ \text{trace } Y = \text{trace } X. \tag{44}$$

Therefore, the matrix Y is similar to X, and they are of the same type of matrix. Then, substituting $B = M$(M is one of the solutions of $B = B^{-1}$) and $B = MA$ into Equation (44),

$$YM = MX \quad \text{and} \quad \underline{YMA} = MAX = \underline{MXA}. \tag{45}$$

Comparing the second and third sides, $AX = XA$. In the same way $AY = YA$.

However, since the coordinate transformation matrix A and the figure transformation matrices X and Y are commutative, then the matrices X and Y have a common relative invariant function $\phi(p)$ from Equation (7) and (23) as

$$\phi(Xp) = \det X\, \phi(p) = \phi(Yp) = \det Y\, \phi(p), \quad \text{where } \phi(Ap) = \phi(p) = -kx^2 + y^2. \tag{46}$$

Thus we conclude that any combination of the matrices A, X, and Y creates a commutative transformation group. Furthermore, any combination of the matrices B, A, X, and Y creates a transformation group based on the orbit of invariant function $\phi(p)$ on both sides of a plane. This super group geometry involves Euclidean geometry and Minkowski plane geometry. An example is shown in Figure A1. of Appendix A.

On the other hand, based on the sign of k, we obtain the existing direction of fold and isotropic lines that vary on the coordinate system. Some cases are presented as follows:

(1) If $k < 0$, then matrix A and the invariant function $\phi(p) = -kx^2 + y^2$ are an elliptic type, and from Equation (27), we can express the matrix $B = MA$ given by

$$M = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix},\ B = MA = \begin{pmatrix} -a & -b \\ c & a \end{pmatrix} = \begin{pmatrix} -\cos\theta & -\sin\theta/\sqrt{-k} \\ -\sqrt{-k}\sin\theta & \cos\theta \end{pmatrix},\ \det B = -1. \tag{47}$$

From Equation (33), we have the fold line f:

$$y = \frac{-c}{a-1}x = \sqrt{-k}\frac{\sin\theta}{\cos\theta - 1}\, x = \sqrt{-k}\cot\frac{\theta}{2}\cdot x = ux. \tag{48}$$

From Equation (34), we have the isotropic line g:

$$y = \frac{-c}{a+1}x = \sqrt{-k}\frac{\sin\theta}{\cos\theta + 1}\, x = \sqrt{-k}\tan\frac{\theta}{2}\cdot x = vx. \tag{49}$$

The existing directions of lines f and g are

$$-\infty < u < \infty,\ -\infty < v < \infty. \tag{50}$$

We observe that an oblique reflection plane made of a fold line f and an isotropic line g exists in all directions centered around the origin. Similar is the case of a negative solution of the matrix $-B$. Therefore, we conclude that this symmetry plane made of oblique reflection planes is completely isotropic**.** This type of plane which has the oblique reflection matrix B with $k < 0$ fits in the space × space type plane and forms an elliptic type plane geometry. When $k = -1$, then Euclidean geometry is given as shown in section 5.3.

(2) If $k > 0$, then matrix A and the invariant function $\phi(p) = -kx^2 + y^2$ is a hyperbolic type, and from Equation (28), we can express the matrix $B = MA$ given by

$$B=\begin{pmatrix}-a & -b\\ c & a\end{pmatrix}=\begin{pmatrix}-\cosh\theta & -\sinh\theta/\sqrt{k}\\ \sqrt{k}\sinh\theta & \cosh\theta\end{pmatrix},\ \det B=-1,\ a=\cosh\theta,\ c=\sqrt{k}\sinh\theta. \tag{51}$$

From Equation (33), we have the fold line f :

$$y=\frac{-c}{a-1}x=-\sqrt{k}\frac{\sinh\theta}{\cosh\theta-1}x=-\sqrt{k}\coth\frac{\theta}{2}\cdot x=ux. \tag{52}$$

From Equation (34), we have the isotropic line g :

$$y=\frac{-c}{a+1}x=-\sqrt{k}\frac{\sinh\theta}{\cosh\theta+1}x=-\sqrt{k}\tanh\frac{\theta}{2}\cdot x=vx. \tag{53}$$

The asymptote line is

$$y=\pm\sqrt{k}x.\ \ (\text{for } \theta\rightarrow\pm\infty) \tag{54}$$

The existing directions of the lines f are

$$-\infty<u<-\sqrt{k}\ \text{ and }\ \sqrt{k}<u<\infty,\ \text{upper and lower quadrant.} \tag{55}$$

The existing directions of the lines g are

$$-\sqrt{k}<u<\sqrt{k},\ \text{left and right quadrant.} \tag{56}$$

The x_2-y_2 axes are

$$y_2\ \text{axis: } x_2=-ax-by=0 \Leftrightarrow y=-\sqrt{k}\coth\theta\cdot x,\ x_2\ \text{axis: } y_2=cx+ay=0 \Leftrightarrow y=-\sqrt{k}\tanh\theta\cdot x \tag{57}$$

The direction of the fold line f exists in the upper and lower quadrant regions, and the direction of the isotropic line g exists in the left and right quadrant regions on the coordinate system. The directions of axes of y and y_2 are the same upper quadrant regions, but the directions of axes of x and x_2 are the inverse. The inverse relation of g and f is the negative solution of the matrix $-B$. Therefore, we conclude that this symmetry plane made of oblique reflection planes is semi-isotropic, as the time axes are f, y, y_2 and the space axes are g, x, x_2. This type of plane which has the oblique reflection matrix B with $k>0$ fits in the space × time type plane, and forms a hyperbolic type plane geometry. However, when $k=1/c^2$ and $y=t$, we call this hyperbolic type plane geometry a Minkowski spacetime geometry. The constant c represents the speed of light.

5. Expected conclusions

5.1. Conceptual answer to the principle of relativity

$\leftarrow v$ S2

S1 $v\rightarrow$

We think the two inertial systems S1 and S2 move on one line, going away from each other at the speed of v m/sec. In the Figure 3., the space-time axes of inertial coordinate systems S1 and S2 are x_1-t_1 on the front side, and x_2-t_2 on the back side of a Minkowski plane. Make them coincide with their origins. From Equation (28) and substituting $y=t$, we have Lorentz transformation $x_L=ax_1+bt_1$, $t_L=kbx_1+at_1$. The x_L-t_L represents the Lorentz coordinate axis. From the first equation, a is a unitless constant, and b is a velocity constant. As the motion of S2 is represented by $x_1=vt_1$ on the front side, then t_L axis is $x_L=0 \Leftrightarrow x_1=vt_1$ and $v=-b/a$ is deduced. From the second equation, k represents reciprocal of the velocity squared in which we put $k=1/c^2>0$ by convention, and c is a velocity constant. Since $\det L=1$, then we obtain $a=1/\sqrt{1-v^2/c^2}=\gamma$. Thus, the Lorentz transformation and its oblique reflection transformation B are defined as follows:

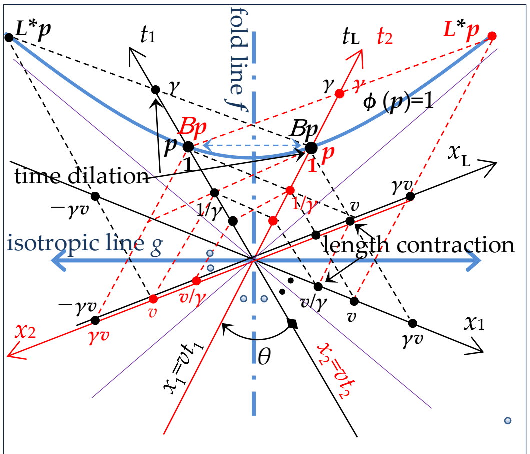

Figure 3. Relativity principle

$$L=\gamma\begin{pmatrix}1 & -v\\ -v/c^2 & 1\end{pmatrix}=MB,\ \det L=1,\ M=\begin{pmatrix}-1 & 0\\ 0 & 1\end{pmatrix}=LB,\ \gamma=1/\sqrt{1-v^2/c^2},$$

$$B=\gamma\begin{pmatrix}-1 & v\\ -v/c^2 & 1\end{pmatrix}=ML=B^{-1},\ \det B=-1,\ p=\begin{pmatrix}0\\1\end{pmatrix},\ p=\begin{pmatrix}0\\1\end{pmatrix},\ Bp=\begin{pmatrix}\gamma v\\ \gamma\end{pmatrix},\ B\begin{pmatrix}0\\ 1/\gamma\end{pmatrix}=\begin{pmatrix}v\\1\end{pmatrix},\ L\begin{pmatrix}0\\1\end{pmatrix}=\begin{pmatrix}-\gamma v\\ \gamma\end{pmatrix},$$

t_2-axis: $x_2=\gamma(-x_1+vt_1)=0$, x_2-axis: $t_2=\gamma(-vx_1/c^2+t_1)=0$.

$$\phi(BLp)=\phi(Lp)=\phi(L^*p)=\phi(Bp)=\phi(p)=\phi\begin{pmatrix}x_1\\t_1\end{pmatrix}=\phi\begin{pmatrix}x_2\\t_2\end{pmatrix}=\phi\begin{pmatrix}x_L\\t_L\end{pmatrix}=-kx^2+t^2,\ k=1/c^2>0,$$

S1→S2: $x_1=vt_1$, S2→S1: $x_2=vt_2$, $x_L=-vt_L$, x_1-axis // Bp—L^*p, isotropic line g // p—Bp.

*Note that Bp or L^*p means figure transformation and Bp or Lp means coordinate transformation.

The fold line f and the isotropic line g of matrix B are drawn perpendicular. It is proposed that the matrix B is the same case of figure 2. as $B=\frac{1}{3}\begin{pmatrix}-5 & 4\\ -4 & 5\end{pmatrix}$ for example, and c=1, v=4c/5, k=1, and $\theta=-1.0986$ which is hyperbolic angle of axis y and y_2 as per Equation (57). The physical symmetry is $x_1=vt_1$ and $x_2=vt_2$, which is supported by the tilt between x_2-t_2 and x_1-t_1 coordinate. If the point p or axis x_L-y_L exists on the front side, then the point Bp or axis x_2-y_2 is hidden on the back side. Because the appearance from inertial system S1 to S2 is the same as from S2 to S1, then the view from the front side is the same as the view from the back side. Any point p on the front side is equivalent to point p on the back side such that p (front)→ Bp (front)→ $B^2p=p$ (back), and vice versa. In the same way, the transformation $p\rightarrow L^*p$ on the front side is equivalent to the transformation $p\rightarrow L^*p$ on the back side. We see time dilation (1→γ sec) and length contraction ($v\rightarrow v/\gamma$ m). Both differences occur by the projection between the axis t_1 and t_2 or x_1 and x_2. Thus, the back and front worlds are completely equivalent in terms of the symmetry of a space × time plane. This means the relativity principle in which any basic law of nature with space and time vectors $\begin{pmatrix}x\\t\end{pmatrix}$ must be oblique reflection B invariant, and also Lorentz transformation invariant, because $L=MB$ and basic laws have x inversion invariance. The two principles are not necessary, but universal limiting velocity c=$1/\sqrt{k}$ is implied on Equation (54), and discussed in my website in section §5.1 in Japanese [5].

5.2. Answer to the arrow of time problem

Why does the physical phenomenon only proceed in one fixed direction of time in spite of the fact that fundamental law of physics has inversion symmetry with respect to time? This is the "arrow of time" problem which has long been unresolved in physics. Physical phenomena occur in linear spacetime in which unit scale intervals are regularly arranged, but its ±direction is not determined. Although time progresses from the past to the future, the direction of time may be positive or negative. Similarly the direction of the eastward straight line can be positive or negative.

To solve this problem, we have to think Minkowski plane from both the back and the front sides, which represents two inertial systems of the same speed. When observing the pendulum motion, if time progresses in the positive direction on the front side and in the negative direction on the back side, then it can be distinguished which side of a plane you and I are on, but this is not the case. Both times go in the negative or in the positive direction according to the future direction. Nature is elaborate so that it cannot be distinguished which side of a plane is the front and which is the back. The symmetry of a space × time plane in spacetime is the heart of the arrow of time problem.

5.3. An example of Euclidean plane

When k=−1, then we obtain that the invariant function is a circle as $\phi(Ap)=\phi(p)=x^2+y^2$, and this is similar to the case of Euclidean geometry, since the coefficients of x and y are equivalent. Meanwhile, we determine that A is a rotation transformation, and B is a reflection transformation. In this case, the relation of a fold line f and an isotropic line g is perpendicular. However, from Equation (47), when rotation angle is $\theta=-\pi/3$ for example, then

$$B=\frac{1}{2}\begin{pmatrix}1 & \sqrt{3}\\ \sqrt{3} & -1\end{pmatrix},\ A=BM=B\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}=\begin{pmatrix}a & b\\ c & a\end{pmatrix}=\begin{pmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{pmatrix}=\frac{1}{2}\begin{pmatrix}1 & -\sqrt{3}\\ \sqrt{3} & 1\end{pmatrix}, \quad (57)$$

where $c = -b = -\sin\theta = \sqrt{3}/2$, $a = \cos\theta = 1/2$.

The fold line f and the isotropic line g are

$$f: y = -\tan\frac{\theta}{2}\cdot x = \tan\frac{\pi}{6}\cdot x = \frac{x}{\sqrt{3}}, \quad g: y = \cot\frac{\theta}{2}\cdot x = -\cot\frac{\pi}{6}\cdot x = -\sqrt{3}x. \tag{58}$$

The x_2-axis and y_2-axis on the back surface are

$$x_2\text{-axis}: y = \frac{c}{a}x = -\tan\ \theta\cdot x = \tan\frac{\pi}{3}\cdot x = \sqrt{3}x, \quad y_2\text{-axis}: y = -\frac{x}{\sqrt{3}}. \tag{59}$$

Considering the front side, the coordinate transformation $\boldsymbol{A}$ has two coordinate axes, namely,

$$x_A\text{-axis}: y = -\sqrt{3}x, \quad \text{and } y_A\text{-axis}: y = x/\sqrt{3}. \tag{60}$$

When $\boldsymbol{X}$ is a figure transformation matrix on the front side, consider any point $\boldsymbol{p}$ is transformed such that the combinations of the matrices $\boldsymbol{B}$, $\boldsymbol{A}$, and $\boldsymbol{X}$ make a closed circuit. For $\boldsymbol{X} = (\det\boldsymbol{X})^{1/2}\boldsymbol{S}$, $\det\boldsymbol{S}=1$, the invariant function is $\phi(\boldsymbol{Ap}) = \phi(\boldsymbol{Bp}) = \phi(\boldsymbol{Sp}) = \phi(\boldsymbol{p}) = x^2+y^2$.

In Figure 4., $0<\det\boldsymbol{X}<1$ is proposed. On the back side, the point $\boldsymbol{Bp}$ is hidden behind $\boldsymbol{p}$ on the front side, and similarly, $\boldsymbol{p}$ is hidden behind $\boldsymbol{Bp}$. The two points of $\boldsymbol{p}$ and $\boldsymbol{p}$ have the same coordinate values, but their coordinate systems are different, x-y axis on the front side, and x_2-y_2 axis on the back side. We regard a coordinate transformation matrix $\boldsymbol{A}$ as a figure transformation matrix $\boldsymbol{A}^{-1}$, in order to easily understand the results of the transformation of point $\boldsymbol{p}$ to make a closed circuit. In Figure 4., matrix $\boldsymbol{A}$ rorates point $\boldsymbol{p}$ to $\boldsymbol{Ap}$ by $\theta=\pi/3$.

- $\boldsymbol{p}$(front surface)→$\boldsymbol{Ap}$(front)→$\boldsymbol{BAp}$(back)→$\boldsymbol{ABAp}=\boldsymbol{Bp}$(back)→$\boldsymbol{BABAp}=\boldsymbol{p}$(front)
- $\boldsymbol{p}$(front surface)→$\boldsymbol{Ap}$(front)→$\boldsymbol{BAp}$(front)→$\boldsymbol{ABAp}=\boldsymbol{Bp}$(front)→$\boldsymbol{BABAp}=\boldsymbol{p}$(front)

Two closed circuits are equivalent to each other. Note that each transformation goes between coordinate systems, either front and front or front and back. Similarly $\boldsymbol{S}$ and $\boldsymbol{X}$ make a closed circuit.

- $\boldsymbol{p}$(front)→ $\boldsymbol{Sp}$(front)→$\boldsymbol{BSp}$(back)→$\boldsymbol{SBSp}=\boldsymbol{Bp}$(back)→$\boldsymbol{BSBSp}=\boldsymbol{p}$(front)
- $\boldsymbol{p}$(front)→ $\boldsymbol{Xp} = (\det\boldsymbol{X})^{1/2}\boldsymbol{Sp}$(front)→ $\boldsymbol{BXp}$(front)→ $\boldsymbol{XBX}=\boldsymbol{Bp}$(front)→$\boldsymbol{BXBXp}=\boldsymbol{p}$(front)
- The solid line ($\boldsymbol{p}\to\boldsymbol{Ap}$ and $\boldsymbol{p}\to\boldsymbol{Xp}$) on the front side is an equivalent transformation to the broken line ($\boldsymbol{p}\to\boldsymbol{Ap}$ and $\boldsymbol{p}\to\boldsymbol{Xp}$) on the back side.

- $\theta = -\angle xOx_2 = \angle xOx_A$
  $= -\angle \boldsymbol{p}O\boldsymbol{Ap} = -\pi/3$
- y-axis // $\boldsymbol{p}$—$\boldsymbol{BAp}$

line g // $\boldsymbol{p}$—$\boldsymbol{Bp}$//$\boldsymbol{Ap}$—$\boldsymbol{BAp}$

- x_2-y_2 is reflected to
  x_A-y_A in x-axis by
  matrix $\boldsymbol{M}$
- Invariant line

$f(\boldsymbol{Bp}) = f(\boldsymbol{p}) = f(\boldsymbol{r}) = (\sqrt{3}x+y)/2$

$\boldsymbol{Bp}+\boldsymbol{p} = 2\boldsymbol{r}$

- Invariant function

$\phi(\boldsymbol{p}) = \phi(\boldsymbol{Ap}) = \phi(\boldsymbol{BAp})$
$= \phi(\boldsymbol{Bp}) = \phi(\boldsymbol{BSp})$
$= \phi(\boldsymbol{Sp})$
$= \phi(\boldsymbol{Ap}) = \phi(\boldsymbol{BAp})$
$= \phi(\boldsymbol{Bp}) = \phi(\boldsymbol{p})$
$= x^2+y^2$

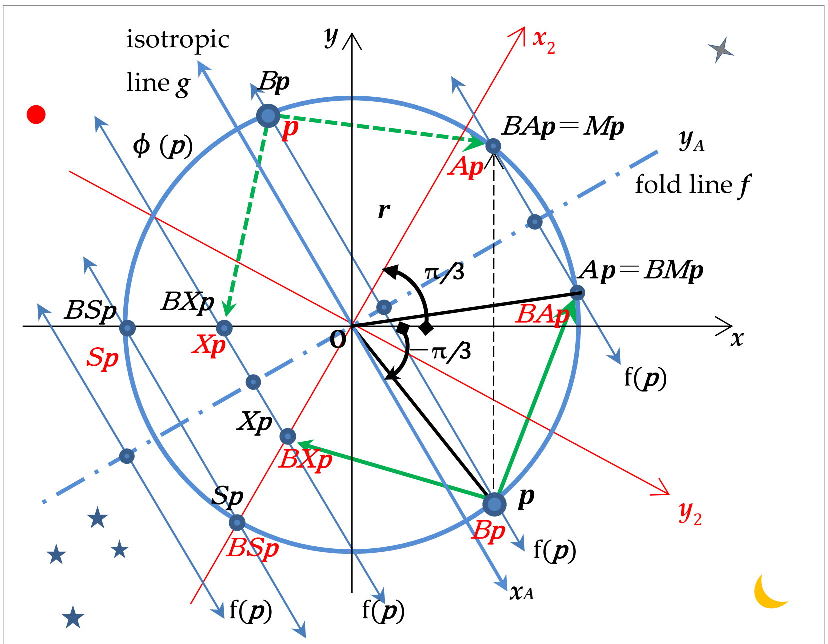

Figure 4. Symmetry transformation on a Euclidean plane

We also observe that the fold line f, the isotropic line g, the invariant lines f($\boldsymbol{p}$), and the invariant function $\phi(\boldsymbol{p}) = x^2+y^2$, have the same shape from each side of the coordinate system. Both sides of the plane are symmetric when compared by viewing the surface from each back and front side. □

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Appendix A Nine-point circle theorem on symmetry planes

Nine–point circle on both Minkowski plane and Euclidean plane hold the same logic.

Point O is the center of circumscribed circle on △ABC.
Point G is the center of gravity of △ABC.
Point H is the crossing point of perpendicular lines of AL,BM,CN.
Point K is the middle point of O and H, and the center of 9-point circle.
Line OGKH is the Euler line.

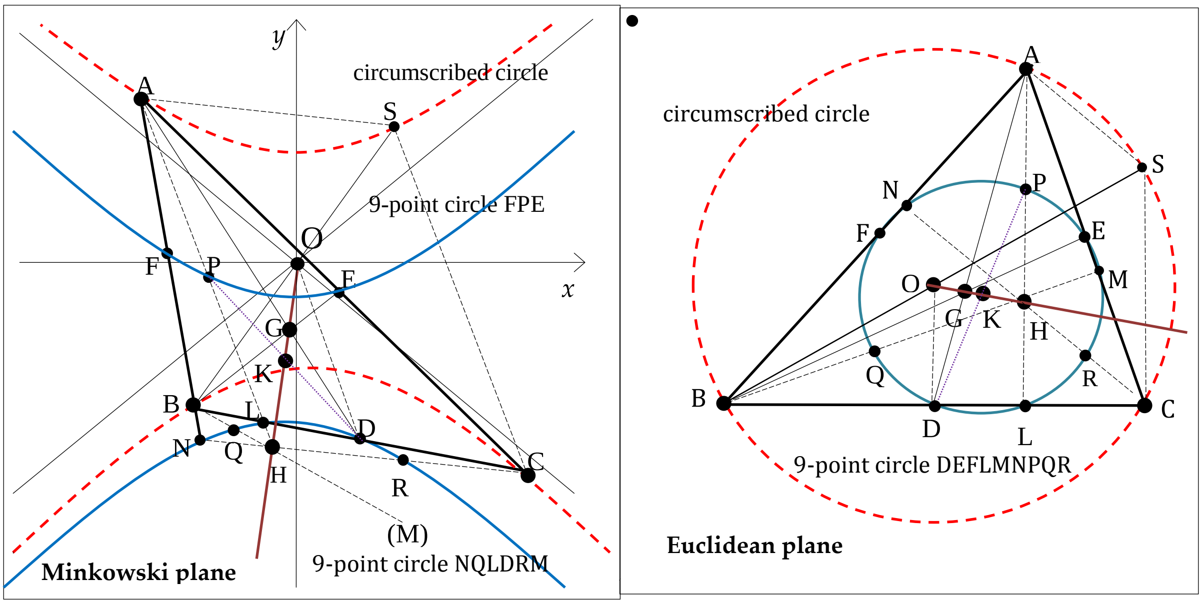

Figure A1. Symmetry plane geometry is constructed and deduced uniformly on both the Euclidean plane and Minkowski plane, which forms the geometry of a super group.

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