Proof of the Relativity Principle

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Abstract: Looking back on the history of the special relativity, Lorentz and Poincaré were on their way to giving a mathematical proof to the result of Michelson-Morley experiment. Meanwhile Einstein published the theory of relativity based on the principle of relativity and the constancy of the speed of light [1]. The problem left by Poincarè in his book "Electricitè et optique"[2] was that a well-developed theory should be able to prove this [relativity] principle very strictly and all at once.

By the way what do you do when you prove that Euclidean geometry holds completely the same on both the front and back symmetric plane? Here is the answer to this problem. By establishing a mathematical derivation of the special relativity principle, my paper gives a positive answer to this problem.

1. Preliminaries

1.1 Terms

- *Spacetime* is a four-dimensional unified entity of space and time without considering all of the matter from the universe. The 4-D *spacetime* is also known as Minkowski space.
- A plane in spacetime is one of two types, namely, *space* × *space* type or *space* × *time* type.

A *space* × *space* type plane is completely isotropic, since the space line is isotropic.
A *space* × *time* type plane is semi-isotropic, since the time line is of unidirectional and the space line is isotropic. This plane is also known as Minkowski plane.

- *Symmetry plane* is a plane with indistinguishable back and front surfaces.

1.2 Relativity principle

Extended Curie's principle

Curie's principle is that in linear physical phenomena, the symmetries of the causes are to be found in the effects. We have extended this principle that in linear space, the symmetry of space is to be derived from the symmetries of subspaces, so is the spacetime transformation. This means that both basic symmetries of isotropy of space and unidirectional of time should be used in order to derive the spacetime transformation.

From this viewpoint, when we review the methodology of the special relativity, the basic symmetry of unidirectional of time which is two different time lines have the same directions of time is not used explicitly in any process on deriving Lorentz transformation, while another basic symmetry of isotropy of space is explicitly used. This means that unidirectional of time is hidden behind the postulates. When we think that the principle of the constancy of the speed of light depends on both the principle of relativity and the Maxwell's law of the speed of light, the remaining independent principle of relativity must be a sufficient but not necessary condition. This might be what Poincaré minded.

Definition of the relativity principle

Let $\boldsymbol{A}=\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be a linear transformation, f($\boldsymbol{p}$) and g($\boldsymbol{q}$) be functions, $\boldsymbol{p}=\binom{x}{y}$ and $\boldsymbol{q}=\binom{u}{v}$ be points on different right-hand coordinate planes, and $\boldsymbol{q}=\boldsymbol{A}\boldsymbol{p}$, f($\boldsymbol{p}$)=f($\boldsymbol{A}^{-1}\boldsymbol{q}$)≡g($\boldsymbol{q}$) by definition hold. If g($\boldsymbol{q}$)=f($\boldsymbol{q}$) =f($\boldsymbol{p}$), then we are not able to distinguish which world of the coordinate plane we are on. This shows how a law f($\boldsymbol{p}$)=f($\boldsymbol{q}$) is born which means the same function f on the different coordinate plane of $\boldsymbol{p}$ and $\boldsymbol{q}$, and this mechanism is called the principle of relativity on physics, which is the same as the idea of Erlangen program on geometry. This mechanism must be subject to the extended Curie's principle. The mathematical definition of the relativity principle is f($\boldsymbol{q}$)= f($\boldsymbol{Ap}$)=f($\boldsymbol{p}$) and det$\boldsymbol{A}$ >0.

1.3 Theorems*

▪ *Invariant line of* 2x2 *matrix* *** These original theorems have proofs in the book [6] or paper [7].**

When the matrix $\boldsymbol{B}=\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ has an eigen value λ=1, it has an invariant line f($\boldsymbol{p}$).

$$\mathrm{f}(\boldsymbol{Bp}) \equiv \mathrm{f}(\boldsymbol{p}) = cx-(a-1)y. \tag{1}$$

▪ *Identity of quadratic invariant function of a* 2x2 *matrix*

Let $\boldsymbol{A}=\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be a matrix, $\boldsymbol{p}=\begin{pmatrix} x \\ y \end{pmatrix}$ be a point,
and $\phi(\boldsymbol{p})=-cx^2+by^2+(a-d)xy$ be a quadratic function.
There exists an identity $\phi(\boldsymbol{Ap}) \equiv \det\boldsymbol{A} \cdot \phi(\boldsymbol{p})$. (2)
If $\det\boldsymbol{A} = 1$, then $\phi(\boldsymbol{Ap}) = \phi(\boldsymbol{p})$.
In this case, $\phi(\boldsymbol{p})$ is called a quadratic invariant function of transformation $\boldsymbol{A}$.

▪ *Polar form of a* 2x2 *special linear matrix*

A special linear matrix $\boldsymbol{S}$ is decomposed using *commutative coefficients k,h* such as

$$\boldsymbol{S}=\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} m+hb & b \\ kb & m-hb \end{pmatrix},$$
where $\det\boldsymbol{S}= m^2-\Delta b^2 = 1$, $\Delta = h^2+k$, $m = (a+d)/2$, $k = c/b$, $2h = (a-d)/b$, $b \neq 0$. (3)

The matrix $\boldsymbol{S}$ has a normalized invariant function such as

$$\phi(\boldsymbol{Sp})=\phi(\boldsymbol{p})=-kx^2+y^2+2hxy= r^2,\ \det\boldsymbol{S}=1,\ \boldsymbol{p}=\begin{pmatrix} x \\ y \end{pmatrix}. \tag{4}$$

This satisfies the definition of the relativity principle.
The matrix $\boldsymbol{S}$ and the invariant function $\phi(\boldsymbol{p})$ is classified into three types based on the sign of the discriminant $\Delta=h^2+k$. If $\Delta<0$, then they are of an elliptic type. If $\Delta>0$, then they are of a hyperbolic type. If $\Delta=0$, then they are of a linear type. Thus we represent a 2x2 special linear matrix $\boldsymbol{S}$ ($\det\boldsymbol{S}= m^2-\Delta b^2 = 1$) by *declination* θ and commutative coefficients *k,h,* using the relations of $\cos^2\theta+\sin^2\theta=1$ or $\cosh^2\theta-\sinh^2\theta=1$.

$$\text{For } \Delta<0,\ \boldsymbol{S}=\boldsymbol{S}(\theta,k,h) = \begin{pmatrix} \cos\theta + \frac{h}{\sqrt{-\Delta}}\sin\theta & \frac{1}{\sqrt{-\Delta}}\sin\theta \\ \frac{k}{\sqrt{-\Delta}}\sin\theta & \cos\theta - \frac{h}{\sqrt{-\Delta}}\sin\theta \end{pmatrix}, \text{ elliptic type.} \tag{5}$$

$$\text{For } \Delta>0,\ \boldsymbol{S}=\boldsymbol{S}(\theta,k,h) = \begin{pmatrix} \cosh\theta + \frac{h}{\sqrt{\Delta}}\sinh\theta & \frac{1}{\sqrt{\Delta}}\sinh\theta \\ \frac{k}{\sqrt{\Delta}}\sinh\theta & \cosh\theta - \frac{h}{\sqrt{\Delta}}\sinh\theta \end{pmatrix}, \text{ hyperbolic type.} \tag{6}$$

$$\text{For } \Delta=0,\ \boldsymbol{S}=\boldsymbol{S}(b,h) = \begin{pmatrix} m+hb & b \\ -h^2b & m-hb \end{pmatrix}, \text{ where } m=\pm1, \text{ linear type.} \tag{7}$$

Any 2x2 non-diagonal regular matrix $\boldsymbol{A}$ is represented by the *polar form* of

$$\boldsymbol{A}= (\det\boldsymbol{A})^{1/2}\,\boldsymbol{S}(\theta,k,h) \text{ or } \boldsymbol{A}= (\det\boldsymbol{A})^{1/2}\,\boldsymbol{S}(b,h). \tag{8}$$

▪ *Extended velocity composition theorem*

We have a velocity composition theorem from Lorentz transformation

$$v_{13}=-v_{31}=(v_{12}+v_{23})/(1+v_{12}v_{23}/c^2).$$

This is equivalently changed to: $(c-v_{12})(c-v_{23})(c-v_{31})=(c+v_{12})(c+v_{23})(c+v_{31})$. (9)

This is extended 3 to n: $(c-v_{12})(c-v_{23})\cdot\cdot\cdot(c-v_{n1})=(c+v_{12})(c+v_{23})\cdot\cdot\cdot(c+v_{n1})$.

2. Introduction

Think everything in a two-dimensional space-time model.

(1) Put right-hand oblique coordinate systems on both face sides of a plane, and make them coincide with their origins. We define a 2x2 back coordinate transformation matrix $\boldsymbol{B}$ as an inside out transformation, then det$\boldsymbol{B}$<0. Turn over the back coordinate transformation $\boldsymbol{B}$ to derive transformation $\boldsymbol{A}$ which transforms between right- hand systems on the front surface.

$$\boldsymbol{A}=\boldsymbol{MB}=(\det\boldsymbol{A})^{1/2}\,\boldsymbol{S}(\theta,k,h),\text{ where } \det\boldsymbol{A}>0,\ \det\boldsymbol{S}(\theta,k,h)=1,\ \boldsymbol{M}=\begin{pmatrix}-1 & 0\\ 0 & 1\end{pmatrix}. \quad (8)$$

The special linear transformation $\boldsymbol{S}(\theta,k,h)$ is given by the polar form. The commutative coefficients k,h are the basis of a plane geometry, because they define the symmetry of a plane as shown later.

(2) When det$\boldsymbol{A}$=1 and transformations $\boldsymbol{S}$ have common commutative coefficients $k,h,$ they form transformation group based on the invariant function of Equation (4) and they create the isometric transformation geometry with the metric of norm $\|\boldsymbol{p}\|=r$.

(3) If the back coordinate transformation $\boldsymbol{B}\neq\boldsymbol{B}^{-1}$, then we are able to distinguish which side of a plane we are on. Therefore, the symmetry plane equation is

$$\boldsymbol{B}=\boldsymbol{B}^{-1},\text{ where } \det\boldsymbol{B}<0. \quad (10)$$

We obtain an *oblique reflection transformation matrix* $\boldsymbol{B}$ with two degrees of freedom:

$$\boldsymbol{B}=\begin{pmatrix}-a & -b\\ c & a\end{pmatrix}=\begin{pmatrix}-a & -b\\ kb & a\end{pmatrix},\ \det\boldsymbol{B}=-1,\ k=c/b,\ \text{eigen values } \lambda=\pm1. \quad (11)$$

(4) Turn over the oblique reflection transformation $\boldsymbol{B}$ to derive transformation $\boldsymbol{F}$ which transforms between right-hand coordinate systems on the front surface,

$$\boldsymbol{F}=\boldsymbol{MB}=\begin{pmatrix}a & b\\ kb & a\end{pmatrix}=\boldsymbol{S}(\theta,k,0),\text{ where } \boldsymbol{M}=\begin{pmatrix}-1 & 0\\ 0 & 1\end{pmatrix},\ \det\boldsymbol{F}=1,\ k=c/b,\ h=0. \quad (16)$$

The matrix $\boldsymbol{F}=\boldsymbol{S}(\theta,k,0)$ is classified by k as follows.
If $k=-1$, then $\boldsymbol{F}$ is referred to as rotation transformation.
If $k=0$, then $\boldsymbol{F}$ is referred to as Galilean transformation.
If $k>0$, then $\boldsymbol{F}$ is referred to as Lorentz transformation.
Thus, we obtain these transformations from symmetry of a plane in spacetime. From Equation (4) and h=0, we have the quadratic invariant function which saticefies the relativity principle,

$$\underline{\phi(\boldsymbol{Fp})=\phi(\boldsymbol{p})}=-kx^2+y^2=r^2,\ \underline{\det\boldsymbol{F}=1}. \quad (20)$$

(5) Symmetry axioms: There are five kinds of symmetry of a plane in spacetime.
a. linearity of a plane in spacetime (homogeneity of spacetime)
This symmetry includes translation of space and time, and reversal of space and time.
b. commutativity of products of linear transformation
c. front-back symmetry of a plane in spacetime

Next two terms are symmetries of a line on a plane.

d. isotropy of space by two space lines which are equivalently inverted on a line

e. unidirectional of time by two time lines having same direction [7]

(6) The commutative coefficients k,h define the symmetry of a plane in spacetime.

symt.	h	k	plane	transformation	created geometry
a	-	-	affine(one sided) pln.	A general linear transf.	affine geometry
ab	h*	k*	front-back asym. pln.	$S(\theta,k,h)$ special linear	asymt. pln. geom.
abc	0	k	front-back symt. pln.	$F=S(\theta,k,0)$ iso-diagonal	symt. pln. geom.
abcd	0	-1	Euclidean plane	$R=S(\theta,-1,0)$ rotation	Euclidean geometry
abcde	0	0	Newtonian plane	$G=S(b,0)$ Galilean	Newtonian mechanc
abcde	0	+	Minkowski plane	$L=S(\theta,k,0)$ Lorentz	relativity principle

* means that coefficients k,h have common values on a front-back asymmetry plane.

(7) Two inertial coordinate systems moving at a constant speed v on a straight line correspond to the right-hand coordinate systems on the front and back Minkowski plane. This is shown in Figure 1 as a conventional form, and in Figure 2 as a symmetry form and both of them shows the same transformation B.

(8) Minkowski plane means that this plane is ruled by Lorentz transformation and holds Minkowski plane geometry with theorems which are formally the same as Euclidean theorems [7]. Euclidean theorems are covariant with respect to rotational transformations because of Equation (20), similar should be the Minkowski plane geometry.

(9) The relativity principle is not a postulate but a property of linear transformation reflecting the basic symmetries of space and time from which this principle is mathematically proved by eigenplane made of two eigenlines belonging to the oblique reflection transformation B.

3. Proof of the Relativity Principle

3.1 Symmetry plane

Put right-hand oblique coordinate systems on both face sides of a plane, and make them coincide with their origins. We define a 2x2 back coordinate transformation matrix B as an inside out transformation, then det B<0. Because it is not distinctive which side of a plane is the back or front, the symmetry plane equation is

$$B=B^{-1} \Leftrightarrow B^2=E, \det B<0. \tag{10}$$

We obtain an oblique reflection transformation matrix B with two degrees of freedom:

$$B=\pm\begin{pmatrix}-a & -b\\ c & a\end{pmatrix}=\pm\begin{pmatrix}-a & -b\\ kb & a\end{pmatrix}, \det B=-1, k=c/b, \text{ eigen values } \lambda=\pm 1. \tag{11}$$

*The signs of variables a, b, and c can be set arbitrarily. Note that it is different from the setting in Eq. (1)

The matrix B has the following properties.

▪ An oblique reflection transformation B guarantees that the front and back coordinate systems are congruent, because their equations of coordinate axes have the same form.

$$\begin{aligned}&\text{Front: } y_2\text{-axis: } x_2=-ax_1-by_1=0,\ \ x_2\text{-axis: } y_2=kbx_1+ay_1=0,\\ &\text{Back: } y_1\text{-axis: } x_1=-ax_2-by_2=0,\ \ x_1\text{-axis: } y_1=kbx_2+ay_2=0.\end{aligned} \tag{12}$$

▪ Since B has an eigen value of 1, then B has an invariant line f($\boldsymbol{p}$) from Equation (1) and putting a→−a,

$$f(B\boldsymbol{p}) \equiv f(\boldsymbol{p}) = cx+(a+1)y. \tag{13}$$

▪ For λ=1 ⇔ a fixed-point equation $B\boldsymbol{p}$=$\boldsymbol{p}$, this eigen line is called a fold line f:

$$cx+(a-1)y = 0. \tag{14}$$

▪ For λ=−1 ⇔ an inversion equation $B\boldsymbol{p}$=−$\boldsymbol{p}$, this eigen line is called an isotropic line g:

$$cx+(a+1)y = 0. \tag{15}$$

This line g is isotropic regarding the origin and is parallel to an invariant line f($\boldsymbol{p}$) given as per Equation (13).

▪ When a point $\boldsymbol{p}$ is in an invariant line f($\boldsymbol{p}$)=s, and the point $\boldsymbol{r}$ is the intersection point of a fold line f and an invariant line f($\boldsymbol{p}$), then in the fold line f, $B\boldsymbol{r}$=$\boldsymbol{r}$, and in the invariant line, f($B\boldsymbol{p}$) =f($\boldsymbol{p}$)=f($\boldsymbol{r}$). Obtained by translating the vector ($\boldsymbol{p}-\boldsymbol{r}$) onto the isotropic line g,

$$B(\boldsymbol{p}-\boldsymbol{r}) =-(\boldsymbol{p}-\boldsymbol{r}) \Leftrightarrow B\boldsymbol{p}-\boldsymbol{r} =-\boldsymbol{p}+\boldsymbol{r} \Leftrightarrow B\boldsymbol{p}+\boldsymbol{p} = 2\boldsymbol{r}.$$

Since the fixed point $\boldsymbol{r}$ is the midpoint between the point $\boldsymbol{p}$ and $B\boldsymbol{p}$, the invariant line f($\boldsymbol{p}$) is isotropic around the fixed point $\boldsymbol{r}$. For the fold line f and the isotropic line g, the inner product of both eigenvectors is

$$\frac{-c}{a-1}\frac{-c}{a+1} = \frac{c^2}{a^2-1} = \frac{c^2}{bc} = \frac{c}{b} = k. \text{ (orthogonal)}$$

Eigenplane is composed of a fold line f and an isotropic line g, and they are generally oblique in figure but orthogonal in equation. →See Figure 1.

The eigenplane belonging to the oblique reflection transformation B is called the *oblique reflection plane*. A point $\boldsymbol{p}$ is transformed to a point $B\boldsymbol{p}$ in a same isotropic line by transformation B, and their middle point is in the fold line.

$$f(B\boldsymbol{p})=f(\boldsymbol{p})=f(\boldsymbol{r}),\ B\boldsymbol{p}+\boldsymbol{p} = 2\boldsymbol{r}.$$

This oblique reflection plane is semi-isotropic centered a fold line. Also a point $\boldsymbol{p}$ on a front surface is transformed to a corresponding point $\boldsymbol{q}$ as $\boldsymbol{q}$=$B\boldsymbol{p}$ on a back surface. The oblique reflection plane is symmetric on the front and back surfaces. (See Figure 1. and 2.)

▪ Turn over the oblique reflection transformation B to derive transformation F which transforms between right-hand coordinate systems on the front surface such as

$$F=MB = \begin{pmatrix} a & b \\ kb & a \end{pmatrix} = S(\theta,k,0), \text{ where } M=\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix},\ \det F=1, k=c/b, h=0, \tag{16}$$
$$\text{eigen values } \lambda=a\pm\sqrt{k}b, \text{ eigen lines } y=\pm\sqrt{k}x,\ k,h \text{ are commutative coefficients.}$$

The transformation F is called a *special iso-diagonal transformation*. This transformation is x-reversal (x_F=−x_2, y_F=y_2) of the back coordinate system x_2-y_2 by the reflection transformation M to form right-hand system x_F-y_F on the front side.

▪ Figure 1. shows the case of B=$\frac{1}{3}\begin{pmatrix} -5 & 4 \\ -4 & 5 \end{pmatrix}$=$B^{-1}$, k=1 on a Minkowski plane and $\boldsymbol{p}$=$\begin{pmatrix} -3 \\ -3 \end{pmatrix}$ for example. Coordinate(x,y) and (x_F,y_F) are on the front side, coordinate (x_2,y_2) is on the back side.

▪ The back coordinate system(x_2-y_2) and the front coordinate system(x-y) are exactly equivalent as shown Equation (12). The fold line f is the midline between y-axis and y_2-axis.

▪ Transformation B has duality of term a. and b.

a. Figure inversion transformation:

$Bp=q=\binom{1}{-1}=\binom{x}{y}$ on the same surface.

b. Back coordinate transformation:

$Bp=q=\binom{1}{-1}=\binom{x_2}{y_2}$ from front to back surface.

$p=Bq=\binom{-3}{-3}$, $p=Bq=\binom{-3}{-3}$,

$r=\binom{-1}{-2}=(p+q)/2\ =\binom{-1}{-2}=r$

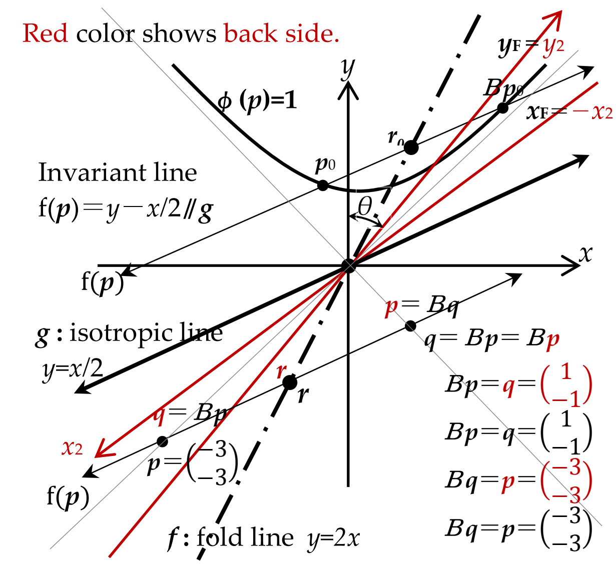

Fig. 1. Oblique reflection plane

Polar form of the special iso-diagonal transformation F

▪ From the polar form of the special linear matrix S, we obtain the polar form of the matrix $F=S(\theta,k,0)$, $h=0$. The θ is the angle of intersection which is called a declination created by both y_1-axis and y_2-axis of the coordinate system.

$$\text{When } k<0,\ F=\begin{pmatrix} a & b \\ kb & a \end{pmatrix}=\begin{pmatrix} \cos\theta & \sin\theta/\sqrt{-k} \\ -\sqrt{-k}\sin\theta & \cos\theta \end{pmatrix}, \text{ where } \theta \text{ is the elliptic angle.} \tag{17}$$

This matrix F is called an elliptic transformation. In particular, when $k=-1$, the matrix F is called a rotational transformation, the matrix B is called a reflection transformation, and they are called orthogonal transformations.

$$\text{When } k>0,\ F=\begin{pmatrix} a & b \\ kb & a \end{pmatrix}=\begin{pmatrix} \cosh\theta & \sinh\theta/\sqrt{k} \\ \sqrt{k}\sinh\theta & \cosh\theta \end{pmatrix}, \text{ where } \theta \text{ is the hyperbolic angle.} \tag{18}$$

This matrix F is called the Lorentz transformation.

$$\text{When } k=0,\ F=\begin{pmatrix} a & b \\ kb & a \end{pmatrix}=\begin{pmatrix} a & b \\ 0 & a \end{pmatrix},\ a=\pm 1. \tag{19}$$

This matrix F is called the Galilean transformation.

▪ From Equation (3), the special iso-diagonal transformation F is given by $h=0$ on $S \Leftrightarrow F(\theta,k)=S(\theta,k,0)$, because the diagonal element of F is $a=d$. From Equation (4), the transformation F has a quadratic invariant function $\phi\ (p)$ which satisfies the definition of the relativity principle.

$$\underline{\phi\ (Fp)=\phi\ (p)}=-kx^2+y^2=r^2,\ \underline{\det F=1},\ p=\binom{x}{y}. \tag{20}$$

Since the function $\phi\ (p)$ is also obtained from $\binom{u}{v}=F\binom{x}{y}$, then $\phi\ (p)$ is the only invariant function of the transformation F.

▪ The transformation $F(\theta, k)=S(\theta, k, 0)$ rule the symmetry plane transformation, create a commutative special iso-diagonal transformation continuous group based on the commutative coefficient k. Since $B=MF$ and $\phi\ (Mp)=\phi\ (p)=-kx^2+y^2$, the oblique reflection transformation B and the special iso-diagonal transformation F and F^2 have the common invariant function $\phi\ (p)$ and the invariant r as shown below.

$$\phi(Bp)=\phi(M(Fp))=\phi(Fp)=\phi(p)=-kx^2+y^2, \text{ where } M=\begin{pmatrix}-1 & 0\\ 0 & 1\end{pmatrix}, p=\begin{pmatrix}x\\ y\end{pmatrix}$$
$$\phi(F^2p)=\phi(F(Fp))=\phi(Fp)=\phi(p)=-kx^2+y^2.$$

When the commutative coefficient k is fixed to the plane, the matrices F and B have one degree of freedom of θ, and any product of these matrices F and B is closed on the orbit of a common invariant function $\phi(p)$. For example,

$$\begin{aligned}&\phi(BF^2\cdots B^{-1}F^{-1}p)=\phi(F^2\cdots B^{-1}F^{-1}p)=\phi(F\cdots B^{-1}F^{-1}p)\\&=\phi(B^{-1}F^{-1}p)=\phi(BF^{-1}p)=\phi(F^{-1}p)=\phi(p)=-kx^2+y^2.\end{aligned} \tag{21}$$

Therefore these transformations B and F create an isometric continuous transformation group based on the quadratic invariant function $\phi(p)$.

- The oblique reflection transformation (back coordinate transformation) B transforms the point p on the front surface to the corresponding point q on the back surface such as

$$q_1=Bp_1,\ q_2=Bp_2. \tag{22}$$

On the front side, the figure transformation X with right-hand system transforms the point p_1 to p_2. Similarly, on the back side, the figure transformation Y transforms the point q_1 to q_2 such as

$$p_2=Xp_1,\ \det X>0,\ q_2=Yq_1,\ \det Y>0. \tag{23}$$

From these four equations, we obtain

$$q_2=Yq_1=\underline{YBp_1}=Bp_2=\underline{BXp_1}. \tag{24}$$

Since the point p_1 is arbitrary and $B=B^{-1}$, we obtain the following.

$$YB=BX \Leftrightarrow Y=BXB \Leftrightarrow BY=XB,\ \det Y=\det X>0,\ \mathrm{tr}\,Y=\mathrm{tr}X. \tag{25}$$

Thus, the matrices X and Y are similar.
Substituting $B=M$ and $B=MF$ into $YB=BX$ on Equation (25) respectively, and therefore

$$YM=MX,\quad \underline{YM}F=MFX=\underline{MX}F. \tag{26}$$

Comparing the second and third sides, $FX=XF$ and similarly $FY=YF$.
The right-hand coordinate transformation F and the figure transformation X, Y are commutative and have the same commutative coefficient k.

When $\det X=1$, the transformations B, F, X, Y and any product of them have a common quadratic invariant function $\phi(p)$ on both sides of a plane such that

$$\phi(Bp)=\phi(Fp)=\phi(Xp)=\phi(Yp)=\phi(p)=-kx^2+y^2=r^2 \tag{27}$$

as an orbit, and the geometry of the isometric transformation continuous group with the norm $\|p\|=r$ is created.

$$\|Bp\|=\|Fp\|=\|Xp\|=\|Yp\|=\|p\|=r. \tag{28}$$

3.2 What k gives? Three types of a symmetry plane

For one oblique reflection transformation B, there corresponds one special iso-diagonal transformation F and one oblique reflection plane. The declination θ is the angle of intersection between the right-hand coordinate y-axes represented by the transformation F. When the declination θ is a whole real number, the existing region of the fold line f and the isotropic line g those of which represent one oblique reflection plane depends on the positive or negative commutative coefficient k.

(1) When $k< 0$, F is elliptic transformation. The polar form of the transformation $B=MF$ is from Equation (17),

$$B=\begin{pmatrix}-a & -b\\ c & a\end{pmatrix}=\begin{pmatrix}-\cos\theta & -\sin\theta/\sqrt{-k}\\ -\sqrt{-k}\sin\theta & \cos\theta\end{pmatrix},\ b=\frac{1}{\sqrt{-k}}\sin\theta,\quad c=kb. \tag{29}$$

The fold line f and the isotropic line g are obtained from Equations (14) and (15).

$$\begin{gathered}\text{the fold line } f:\ y=\frac{-c}{a-1}x=\sqrt{-k}\frac{\sin\theta}{\cos\theta-1}\ x=\sqrt{-k}\cot\frac{\theta}{2}\cdot x=ux,\\ \text{the isotropic line } g:\ y=\frac{-c}{a+1}x=\sqrt{-k}\frac{\sin\theta}{\cos\theta+1}\ x=\sqrt{-k}\tan\frac{\theta}{2}\cdot x=vx,\\ \text{the slopes of eigen lines } f,g:\ -\infty<u<\infty,\ \infty<v<\infty.\end{gathered} \tag{30}$$

The half angle of declination θ determines the azimuth of the fold and isotropic line. When the declination of elliptic angle θ is a whole real number, the slopes u and v are also whole real numbers. Therefore, this plane is isotropic and thought to be space x space type because an oblique reflection plane is able to be transformed by F equally in all directions around the origin. This plane is called the extended Euclidean plane or elliptic type plane, and extended Euclidean plane geometry holds. In particular, when $k=-1$, the invariant function $\phi(\boldsymbol{p})=x^2+y^2$ is a circle and this plane holds Euclidean geometry.

(2) When $k> 0$, F is Lorentz transformation. The polar form of the transformation $B=MF$ is from Equation (18),

$$B=\begin{pmatrix}-a & -b\\ c & a\end{pmatrix}=\begin{pmatrix}-\cosh\theta & -\sinh\theta/\sqrt{k}\\ \sqrt{k}\sinh\theta & \cosh\theta\end{pmatrix},\ b=\frac{1}{\sqrt{k}}\sinh\theta,\ c=kb. \tag{31}$$

The fold line f and the isotropic line g are obtained from Equations (14) and (15).

$$\begin{gathered}\text{the fold line } f:\ y=\frac{-c}{a-1}x=-\sqrt{k}\frac{\sinh\theta}{\cosh\theta-1}\ x=-\sqrt{k}\coth\frac{\theta}{2}\cdot x=ux,\\ \text{the isotropic line } g:\ y=\frac{-c}{a+1}x=-\sqrt{k}\frac{\sinh\theta}{\cosh\theta+1}\ x=-\sqrt{k}\tanh\frac{\theta}{2}\cdot x=vx.\end{gathered} \tag{32}$$

$$\text{the asymptotic line}:\ y=\pm\sqrt{k}x. \tag{33}$$

Both asymptotes divide the coordinate plane into four quadrants. The regions of existence of the eigen lines f and g are: The slope u of the fold line f is $-\infty<u<-\sqrt{k}$ and $\sqrt{k}<u<\infty$ in the upper and lower quadrants. The slope v of the isotropic line g is $-\sqrt{k}<v<\sqrt{k}$ in the left and right quadrants. When the declination of hyperbolic angle θ is a whole real number, the existence of the fold lines are unevenly distributed in the upper and lower quadrants partitioned by both asymptotes, and a point on the invariant function $\phi(\boldsymbol{p})$, namely upward hyperbola, is transitively transformed across the fold line f to a point on the same hyperbola in the same quadrant by the transformation F. Since the azimuths of both fold lines are aligned to upward, the fold line f is thought to be equivalent to the time line and the isotropic line g is thought to be equivalent to the space line. The plane of this space x time type is semi-isotropic. This plane is called Minkowski plane, and holds hyperbolic type plane geometry, i.e, Minkowski plane geometry.

(3) When $k=0$, F is the Galilean transformation. From Equation (19), we obtain $B=MF=\begin{pmatrix}-a & -b\\ 0 & a\end{pmatrix}$, $a=\pm1$, $\det B=-1$.

$$\begin{gathered}\text{the fold line } f\text{: } x=-by/2 \text{ (when } a=1\text{), the isotropic line } g: y=0\ (x\text{-axis}),\\ \text{the invariant line: } \mathrm{f}(B\boldsymbol{p})=\mathrm{f}(\boldsymbol{p})=y,\end{gathered} \tag{34}$$

the invariant function: $\phi(Bp) = \phi(Fp) = \phi(p) = y^2$.

When y=t (time), time is an invariant quantity from the invariant function $\phi(Fp) = \phi(p) = t^2$, which is consistent with Newton's idea of absolute time, so this is called a Newtonian plane. This plane is semi-isotropic, and the spatial axis (y=0) of the oblique reflection planes is shared.

From the above, three types of the symmetry plane are extended Euclidean plane and Minkowski plane and Newtonian plane. The geometries of these planes satisfy the extended Curie's principle, and also the relativity principle from Equation (20). Thus, the laws on the symmetry plane must be made from the invariant function $\phi(p)$ with the norm $\|p\|$=r and they are obviously the special iso-diagonal transformation F invariant.

[End of proof]

3.3 Symmetric spacetime structure of two inertial coordinate systems

Consider two inertial coordinate systems S1 and S2 moving away from each other on a line with constant velocity v. Both systems are equivalent. When each of the two inertial systems sees the other from a space x time right-hand system, the velocity and the positive orientation of the space axis are reversely related to each other, so the two systems correspond to the two sides of the Minkowski plane. The origins of both inertial coordinate systems are made to coincide. In the Figure 2., the coordinate axis x_1-t_1 of S1 is on the front side and x_2-t_2 of S2 is on the back side (red color). The back coordinate transformation is the oblique reflection transformation B. On the Minkowski plane, the special iso-diagonal transformation F is Lorentz transformation $L(=MB)$ which transforms from coordinate system S1 to Lorentz coordinate system S_L such that

$$\begin{pmatrix} x_2 \\ t_2 \end{pmatrix} = B\begin{pmatrix} x_1 \\ t_1 \end{pmatrix} \text{ where } B = \begin{pmatrix} -a & -b \\ kb & a \end{pmatrix}, \text{ multiplying both sides by } M = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix},$$

$$M\begin{pmatrix} x_2 \\ t_2 \end{pmatrix} = MB\begin{pmatrix} x_1 \\ t_1 \end{pmatrix}, \text{ therefore } \begin{pmatrix} -x_2 \\ t_2 \end{pmatrix} = \begin{pmatrix} x_L \\ t_L \end{pmatrix} = L\begin{pmatrix} x_1 \\ t_1 \end{pmatrix} \text{ where } L = MB = \begin{pmatrix} a & b \\ kb & a \end{pmatrix}. \quad (35)$$

The expansion formula for the Lorentz transformation is

$x_L = ax_1 + bt_1$, $t_L = kbx_1 + at_1$, where k>0.

From the first equation, a is a unitless constant and b is a velocity constant. In the x_1-t_1 coordinate system, the motion of S2 is expressed as $x_1 = vt_1$, while the expression for the t_L axis is $x_L = ax_1 + bt_1 = 0$, so $v = x_1/t_1 = -b/a$. In the second equation, k is the inverse of the velocity squared, and by convention we use the velocity constant c as $k = 1/c^2$. As $\det L = 1$, so we obtain $a = (1 - v^2/c^2)^{-1/2} = \gamma \geqq 1$. Thus, from the two inertial coordinate systems with velocity v, the Lorentz transformation L and its oblique reflection transformation B are specifically obtained on Equation (36).

Figure 2. shows the same $B = \frac{1}{3}\begin{pmatrix} -5 & 4 \\ -4 & 5 \end{pmatrix}$ as Figure 1. with k=c=1, v=4/5, γ= 5/3.

Inertial frame 1: S1(x_1,t_1), Inertial frame 2: S2(x_2,t_2), Lorentz cord. sys.: SL(x_L,t_L)

Rewrite Figure 1. with the fold line and isotropic line as cross centerlines to obtain Figure 2, but replace from y axis to t axis. Fold line f : $t = 2x$, isotropic line g : $t = x/2$. $L\begin{pmatrix} v \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1/\gamma \end{pmatrix}$, $L\begin{pmatrix} \gamma v \\ \gamma \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$,

$$L\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \gamma\begin{pmatrix} 1 \\ -v/c^2 \end{pmatrix}, \ B\begin{pmatrix} v \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1/\gamma \end{pmatrix}, f\!: B\begin{pmatrix} 3 \\ 6 \end{pmatrix} = \begin{pmatrix} 3 \\ 6 \end{pmatrix}, \ g\!: B\begin{pmatrix} 6 \\ 3 \end{pmatrix} = \begin{pmatrix} -6 \\ -3 \end{pmatrix}, \quad x=ct: B\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -1/3 \\ 1/3 \end{pmatrix}, \ L\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1/3 \\ 1/3 \end{pmatrix}.$$

$$L = \gamma\begin{pmatrix} 1 & -v \\ -v/c^2 & 1 \end{pmatrix}, \ B = \gamma\begin{pmatrix} -1 & v \\ -v/c^2 & 1 \end{pmatrix} = ML, \ \det B = -1, \ L\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \gamma\begin{pmatrix} -v \\ 1 \end{pmatrix},$$
$$L^{-1} = \gamma\begin{pmatrix} 1 & v \\ v/c^2 & 1 \end{pmatrix}, \ L^{-1}\begin{pmatrix} x_L \\ t_L \end{pmatrix} = \begin{pmatrix} x_1 \\ t_1 \end{pmatrix}, \ \gamma = 1/\sqrt{1 - v^2/c^2}, \text{ c is a velocity constant.} \quad (36)$$

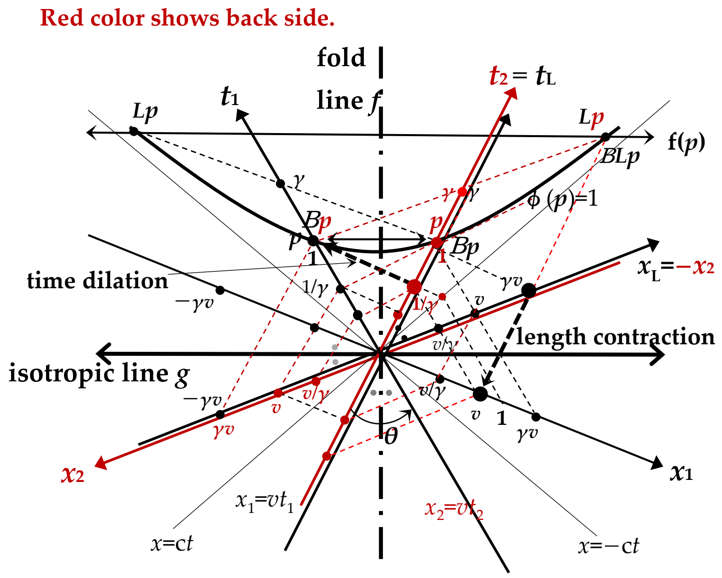

Figure 2. Minkowski plane and two inertial frames

Any two inertial systems on a straight line are drawn as symmetrical as Figure 2.

Maximum universal speed c

From Equation (9), we obtain the limit speed of inertial frames, where c is a velocity constant.

$$(c-v_{12})(c-v_{23})\cdots(c-v_{n1})=(c+v_{12})(c+v_{23})\cdots(c+v_{n1}) \quad (9)$$

$$\Leftrightarrow (c-v_{12})^2(c-v_{23})^2\cdots(c-v_{n1})^2=(c^2-v_{12}^2)(c^2-v_{23}^2)\cdots(c^2-v_{n1}^2)\geqq 0,$$

and n is integer $\Leftrightarrow c^2-v_{ij}^2\geqq 0 \Leftrightarrow (v_{ij}-c)(v_{ij}+c)\leqq 0$, therefore $-c\leqq v_{ij}\leqq c$. (37)

where v_{ij} is the speed of inertial frame from i to j.

Any speed of inertial frame v_{ij} is not able to exceed the velocity constant c which means the maximum universal speed. From Maxwell's law, the speed of light in the inertial reference frame is constant and the maximum speed in nature is the light speed in vacuum. On the basis (h=0, k=1/c^2) of the Minkowski plane that guarantees the front-back symmetry in the empty spacetime, the maximum universal velocity constant c must be the speed of light in vacuum. This is equivalent to applying the principle of relativity to Maxwell's law of the speed of light.

4. Conclusion

It is noted that any function f(p) held on the front side holds the same as function f(p) on the back side such as

Back coordinate transformation $Bp=q$, figure inversion transformation $Bq=p$,

$$B=B^{-1},\ \underline{f(p)}=f(B^{-1}q)=f(Bq)=\underline{f(p)}.$$

This symmetry is limited only between front and back surfaces as the transformation B plays the role of intermediary in the front-back symmetry.

On the other hand the invariant function $\phi(Fp)=\phi(p)$ satisfies the relativity principle, because it keeps the same function of $\phi(p)=\phi(q)$, $q=Fp$ on the different coordinate systems of $p=\binom{x}{y}$ and $q=\binom{u}{v}$ by the transformation F such that

$$\underline{\phi(Fp)=\phi(p)}=-kx^2+y^2=r^2,\ \underline{\det F=1}. \quad (20)$$

Also this transformation F satisfies the extended Curie's principle as shown in section

3.2. Thus, the only invariant function $\phi(\boldsymbol{p})$ of transformation $\boldsymbol{F}$ should be the core of the law of a symmetry plane. Therefore, the law of spacetime must be the variations of invariant function $\phi(\boldsymbol{p})$ with its invariant r such that

$$\phi(\boldsymbol{F}(\boldsymbol{p}_1 \pm \boldsymbol{p}_2)) = \phi(\boldsymbol{p}_1 \pm \boldsymbol{p}_2) \text{ or } \phi(\boldsymbol{F}\frac{d}{dr}\boldsymbol{p}) = \phi(\frac{d}{dr}\boldsymbol{p}) \text{ or } \phi(\boldsymbol{F}\int \boldsymbol{p}\,\mathrm{d}r) = \phi(\int \boldsymbol{p}\,\mathrm{d}r). \quad (38)$$

We are able to find these variations in the basic laws of physics on Minkowski plane or Newtonian plane and theorems of geometry on Euclidean plane or Minkowski plane based on Equations (27) and (28) [6]. For example, the inner product of the symmetry plane is derived from $\phi(\boldsymbol{F}\boldsymbol{p}) = \phi(\boldsymbol{p}) = -kx^2+y^2 = \|\boldsymbol{p}\|^2 = (\boldsymbol{p}, \boldsymbol{p})$.

$$(\phi(\boldsymbol{F}\boldsymbol{p}_1) + \phi(\boldsymbol{F}\boldsymbol{p}_2) - \phi(\boldsymbol{F}(\boldsymbol{p}_1 - \boldsymbol{p}_2)))/2 = (\phi(\boldsymbol{p}_1) + \phi(\boldsymbol{p}_2) - \phi(\boldsymbol{p}_1 - \boldsymbol{p}_2))/2 = -kx_1x_2 + y_1y_2 = (\boldsymbol{p}_1, \boldsymbol{p}_2).$$

My paper establishes the followings. On the basis of the structure of a linear plane, from the symmetry of the space x space type plane (Euclidean plane), we have the reflection and the rotation transformation group which form Euclidean geometry.
Also, from the symmetry of the space x absolute time type plane (Newtonian plane), we have the oblique reflection and the Galilean transformation group which form Newtonian mechanics.
Finally, from the symmetry of the space x time type plane (Minkowski plane), we obtain the oblique reflection and the Lorentz transformation group which form the relativity principle. Therefrom, Minkowski plane geometry and the special theory of relativity are established [2][3].

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