Geometric Structure of Spacetime Plane and Relativity Principle

2019.11.30　　Hiroaki Fujimori

abstract

Euclidean geometry, inherited from ancient Greece, was modeled on axiomatic methods in modern science. Hilbert's "Foundations of Geometry"[1] supplemented the lacking axioms, and seemed to have reached the stage of completion as plane geometry, but still questioned why the axiom system did not depart from the properties of a plane itself[2]. Looking back on the history of special relativity, Lorentz and Poincaré were on their way to give a mathematical proof to the results of the Michelson-Morley experiment[3]. Meanwhile Einstein published the special theory of relativity[4] based on the invariant speed of light. At a glance, all things had been well done by this theory, but it is not enough. Digging into why the relativity principle holds, we arrived at a deeper symmetry of spacetime. A paragraph of Hannyashin Sutra "空即是色：Kuu soku ze shiki" is interpreted as "emptiness contains form of cosmos". From the viewpoint of spacetime substantivalism, empty space is treasure trove to discover the hidden rule of the cosmos. In spacetime, there are lots of geometric objects such as lines, planes, spheres, and inertial coordinate systems, etc. These objects have an individual geometry on them and also internal harmony between them. Considering the ways of Heaven, we observed that the fundamental symmetry in the cosmos is a plane with back and front indistinguishable surface in which the basic laws of nature with position vectors must be subject to this symmetry.

How to prove that "Euclidean plane is inverse invariant for any line on itself" ?

【Put an origin on any point in Euclidean plane. The rotatiom matrix A and the reflection matrix B are established on Euclidean plane as $A=\begin{pmatrix}\cos\theta & \sin\theta \\ -\sin\theta & \cos\theta\end{pmatrix}$ and $B=\begin{pmatrix}\cos\theta & -\sin\theta \\ -\sin\theta & -\cos\theta\end{pmatrix}$.

It is noted that $B=B^{-1}$ ⇔ $B^2=E$, $A=BM$, where $M=\begin{pmatrix}1 & 0 \\ 0 & -1\end{pmatrix}$, $E=\begin{pmatrix}1 & 0 \\ 0 & 1\end{pmatrix}$.

The reflection matrix B transforms a point $\boldsymbol{p}$ on the front side to correspondent rear point $\boldsymbol{q}$ on the back side as $\boldsymbol{q}=B\boldsymbol{p}$, and also to reflection point $\boldsymbol{q}$ on the front side as $\boldsymbol{q}=B\boldsymbol{p}$. The coordinate axis x_2-y_2 is on the back side, and x-y is on the front side. From the reflection matrix B, there are relations as follows.

y_2 axis : $x_2=\cos\theta\cdot x-\sin\theta\cdot y=0$ ⇔ $y=\cot\theta\cdot x=x/\sqrt{3}$, where $\theta=\pi/3$ for example and

x_2 axis : $y_2=-\sin\theta\cdot x-\cos\theta\cdot y=0$ ⇔ $y=-\tan\theta\cdot x=-\sqrt{3}x$.

The eigen values, eigen lines through the origin, and eigen plane of matrix B are given by

- Eigen values are $\lambda=\pm1$, as trace $B=0$ and det $B=-1$.
- For $\lambda=1$ ⇔ Fixed point equation $B\boldsymbol{p}=\boldsymbol{p}$, then this eigen line is named fold line f :

$$\cos\theta\cdot x-\sin\theta\cdot y=x \iff y=\frac{\cos\theta-1}{\sin\theta}\cdot x=-\tan\frac{\theta}{2}\cdot x=-x/\sqrt{3} \qquad (0)$$

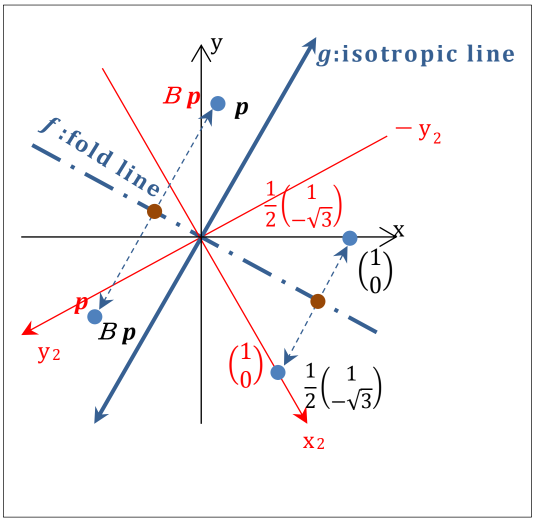
- For $\lambda=-1$ ⇔ Inverse equation $B\boldsymbol{p}=-\boldsymbol{p}$, then this eigen line is named isotropic line g :

$$\cos\theta\cdot x-\sin\theta\cdot y=-x \Leftrightarrow y=\frac{\cos\theta+1}{\sin\theta}\ x=\cot\frac{\theta}{2}\cdot x=\sqrt{3}x$$

Line segment ($\boldsymbol{p}-B\boldsymbol{p}$) is parallel to line g and its middle point is in the line f.

The point $\boldsymbol{p}$ on front side is transformed by B as $\boldsymbol{p}(\text{front})\rightarrow B\boldsymbol{p}(\text{front})\rightarrow B^2\boldsymbol{p}=\boldsymbol{p}(\text{back})$.

The point $\boldsymbol{p}$ on back side is transformed by B as $\boldsymbol{p}(\text{back})\rightarrow B\boldsymbol{p}(\text{back})\rightarrow B^2\boldsymbol{p}=\boldsymbol{p}(\text{front})$.



Example $\theta=\pi/3$, $\tan\theta=\sqrt{3}$, $\tan\frac{\theta}{2}=1/\sqrt{3}$

$$B=\begin{pmatrix}\cos\theta & -\sin\theta\\ -\sin\theta & -\cos\theta\end{pmatrix}=\frac{1}{2}\begin{pmatrix}1 & -\sqrt{3}\\ -\sqrt{3} & -1\end{pmatrix}$$

front side | **back side**

$\boldsymbol{p}=\begin{pmatrix}1\\0\end{pmatrix}$ ▪ ⇔ ▪ $B\boldsymbol{p}=\frac{1}{2}\begin{pmatrix}1\\-\sqrt{3}\end{pmatrix}$

↓ ↑ ↑ ↓

$B\boldsymbol{p}=\frac{1}{2}\begin{pmatrix}1\\-\sqrt{3}\end{pmatrix}$ ▪ ⇔ ▪ $\boldsymbol{p}=\begin{pmatrix}1\\0\end{pmatrix}=B^2\boldsymbol{p}$

$\boldsymbol{p}\rightarrow B\boldsymbol{p}\rightarrow\mid\rightarrow B^2\boldsymbol{p}=\boldsymbol{p}$

$B^2\boldsymbol{p}=\boldsymbol{p}\leftarrow\mid\leftarrow B\boldsymbol{p}\leftarrow\boldsymbol{p}$

Therefore the point $\boldsymbol{p}$ on the front side is equivalent to the point $\boldsymbol{p}$ on the back side, so the plane is symmetric. Since the direction of a hold line f can be in all directions from eq(0), then "Euclidean plane is inverse invariant for any line on itself".□】

The inverse proposition that "If a plane is symmetric, then we have an Euclidean plane" is partialy true . The equation $B=B^{-1}$ means that the plane is symmetric.

What does the symmetry of a plane deduce?

Put oblique right-hand coordinate systems on both face sides of a plane, and make them coinside with their origines. We define 2x2 rear surface coordinate transformation matrix B as inside out transformation, then $\det B<0$. Because it is indistinguishable which side of a plane is back or front, the symmetric plane equation is $B=B^{-1}\ \Leftrightarrow\ B^2=E$.

Solving this equation, we obtained **an oblique reflection transformation** B.

$$B=\begin{pmatrix}a & -b\\ c & -a\end{pmatrix}=\begin{pmatrix}a & -b\\ kb & -a\end{pmatrix},\quad \det B=-1,\ k=c/b, \text{ and } B \text{ has two degree of freedom.}$$

We derive the matrix $A=BM=\begin{pmatrix}a & b\\ c & a\end{pmatrix}=\begin{pmatrix}a & b\\ kb & a\end{pmatrix}$, where $M=\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$, $\det A=a^2-kb^2=1$.

The matrix A is a coordinate transformation between right-hand systems in which y-coordinate on the rear side is reflected in x-axis by matrix M. It is known that if $k=-1$, then the matrix A is a rotation transformation, and matrix B is a reflection transformation. If $k<0$, $k=0$, and $k>0$ in order, then the matrix A is called elliptic transformation, Gallilean transformation and Lorentz transformation. When we fix commutative coefficient k, then matrices A and B make isometric transformation group.(→eq(42))

Thus, the symmetry of a plane gives rise to Euclidean geometry based on an authogonal transformation group of A and B when $k=-1$, and the principle of relativity when $k>=0$.

1. Terms, Definitions, Axioms, and Mathematical Preparations

Terms :

- **An oblique reflection plane** has a fold line and isotropic lines. A point $\boldsymbol{p}$ is transformed to a point $B\boldsymbol{p}$ in the same isotropic line by a transformation matrix B, and their middle point is in the fold line. (→see Figure1)

Definitions :

- **Empty spacetime** is a 4-dimensional unified entity of space and time without considering all the matter from the universe. In this paper, the word “empty” is abbreviated.

  Space is continuous, infinite, homogeneous, 3-dimensional, and isotropic.

  Time is continuous, infinite, homogeneous, 1-dimensional of one direction, and irreversible.
- **A line** in spacetime is 1-dimensional, and it is inverse invariant for any point on itself.
- **A plane** in spacetime is 2-dimensional, and it is inverse invariant for any axis of a fold line passing through any two points on itself.
- **An asymmetric plane** is a plane with back and front distinguishable surface.
- **A symmetric plane** is a plane with back and front indistinguishable surface.
- A line of coordinate axis on a plane in spacetime is of two types, one is **space line**, and the other is **time line**. Space line is isotropic, and time line is of one direction.
- A coordinate system on a plane in spacetime is of two types, one is **space*space type** and the other is **space*time type**.

  Space*space type plane is completely isotropic, since the space line is isotropic. This type of plane exists as subspace of a 3-dimensional space in an inertial system.

  Space*time type plane is semi-isotropic, since the time line is of one direction and the space line is isotropic. This type of plane exists when we think one dimensional space in which all inertial systems move on one line and each inertial cooedinate system with space-axis and time-axis coexists on one common space*time plane.
- **Invariant function** : Let $A=\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be a matrix, $\boldsymbol{p}=\begin{pmatrix} x \\ y \end{pmatrix}$ be a point, and f be a function of $\boldsymbol{p}$. If $f(A\boldsymbol{p})=f(\boldsymbol{p})$, then this function $f(\boldsymbol{p})$ is called an invariant function of A.

Axioms :

- **Inertial system axiom** : There are infinite empty inertial coordinate systems(or inertial systems) in empty spacetime. Each has its own 4-dimensional spacetime and keeps its uniform motion on a straight line.
- **Symmetric plane axiom** : It is indistinguishable which side of a plane is back or front.

Mathematical Preparations[5] **:**

▪ **Invariant line** f($\boldsymbol{p}$) of a 2×2 matrix $B=\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is the solution of a first order invariant function equation of the form : $f(B\boldsymbol{p}) \equiv f(\boldsymbol{p}) = ux+vy \Leftrightarrow u(ax+by)+v(cx+dy)=ux+vy$

$\Leftrightarrow [(a-1)u+cv]x+[bu+(d-1)v]y=0 \Leftrightarrow$ x,y are arbitrary and in order to have a non-self-planatory solution of u,v, the determinant=$(a-1)(d-1)-bc=0$ (1)

When the matrix B has an eigen value $\lambda=1$, we obtained an invariant line f($\boldsymbol{p}$) by substituting $u=c$, $v=-(a-1)$ on $f(\boldsymbol{p})=ux+vy$, thus $f(B\boldsymbol{p}) \equiv f(\boldsymbol{p}) = cx-(a-1)y$. (2)

▪ **Quadratic invariant function** $\phi(\boldsymbol{p})$ of a 2×2 matrix $A=\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is the solution of a second order invariant function equation of the form : $\phi(A\boldsymbol{p}) \equiv \phi(\boldsymbol{p}) = ux^2+vy^2+wxy$ (3)

$\Leftrightarrow \phi(A\boldsymbol{p}) = \phi(ax+by,\ cx+dy) = u(ax+by)^2+v(cx+dy)^2+w(ax+by)(cx+dy) = ux^2+vy^2+wxy$

$\Leftrightarrow [(a^2-1)u+c^2v+acw]x^2+[b^2u+(d^2-1)v+bdw]y^2+[2abu+2cdv+(ad+bc-1)w]xy=0$

Since x^2, y^2, and xy are arbitrary, each coefficient must be equal to 0, and we obtained a simultaneous equation of u,v, and w given by

$$\begin{pmatrix} a^2-1 & c^2 & ac \\ b^2 & d^2-1 & bd \\ 2ab & 2cd & ad+bc-1 \end{pmatrix}\begin{pmatrix} u \\ v \\ w \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}. \qquad (4)$$

In order to have solutions other than self-planatory solution of $u=v=w=0$, the determinant of the 3x3 matrix on eq(4) must be equal to zero.

Thus, determinant=$(ad-bc-1)[(ad-bc+1)^2-(a+d)^2]=0$. (5)

Here, by putting a solution of $u=-c$, $v=b$, $w=a-d$, we obtained an **identity** :

$\phi(A\boldsymbol{p}) \equiv \det A \cdot \phi(\boldsymbol{p})$, where A is a matrix $A=\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $\boldsymbol{p}$ is a point $\boldsymbol{p}=\begin{pmatrix} x \\ y \end{pmatrix}$, and $\phi(\boldsymbol{p})$ is a quadratic function given by $\phi(\boldsymbol{p})=-cx^2+by^2+(a-d)xy$. (6)

(1) From eq(5) and eq(6), if $\det A=1$, then $\phi(A\boldsymbol{p})=\phi(\boldsymbol{p})=-cx^2+by^2+(a-d)xy$. (7)

In this case, $\phi(\boldsymbol{p})$ is the second order **invariant function** of the matrix A.

$\phi(\boldsymbol{p})$ is allowed to be multiplied by a scale factor, such that

if $\Phi(\boldsymbol{p})=r\phi(\boldsymbol{p})$, then $\underline{\Phi(A\boldsymbol{p})}=r\phi(A\boldsymbol{p})=r\det A\ \phi(\boldsymbol{p})=r\phi(\boldsymbol{p})=\underline{\Phi(\boldsymbol{p})}$. (8)

(2)We also obtained another invariant function from eq(5).

If $\det A=-1$ and $\mathrm{tr}A=a+d=0 \Leftrightarrow$ if eigen values of A are $\lambda=\pm1$, then we changed the notation of matrix A into B, and the invariant function has the same part of eq(7) when the cross term of xy is eliminated such that

when $B=\begin{pmatrix} a & -b \\ c & -a \end{pmatrix}$, $\det B=-1$, $\lambda=\pm1$, then $\phi(B\boldsymbol{p})=\phi(\boldsymbol{p})=-cx^2+by^2$. (10)

(3)Meanwhile, if $\det A \neq \pm1$, then $\phi(\boldsymbol{p})$ is the **relative invariant function** of the matrix A.

▪ A special linear transformation S has commutative coeffcients k,h and is disassembled as

$S=\begin{pmatrix} a & b \\ c & d \end{pmatrix}=\begin{pmatrix} m+hb & b \\ kb & m-hb \end{pmatrix}$, where $\det S=m^2-\varDelta b^2=1$, $\varDelta=h^2+k$, $m=(a+d)/2$, $k=c/b$, $2h=(a-d)/b$, and $b\neq0$. (11)

When $S_1=\begin{pmatrix} m_1+hb_1 & b_1 \\ kb_1 & m_1-hb_1 \end{pmatrix}$, $S_2=\begin{pmatrix} m_2+hb_2 & b_2 \\ kb_2 & m_2-hb_2 \end{pmatrix}$, then matrices S_1, S_2 and S_1S_2 have common commutative coefficients k,h, and $S_1S_2=S_2S_1$. (12)

From eq(7) and eq(8), we obtained that matrix S has a normalized invariant function :

$$\phi\,(S\boldsymbol{p})=\phi\,(\boldsymbol{p})=-kx^2+y^2+2hxy. \tag{13}$$

We classified the matrix S and the invariant function $\phi\,(\boldsymbol{p})$ three types based on the sign of the discriminant $\Delta=h^2+k$.

If $\Delta<0$, then they are of elliptic type.

If $\Delta>0$, then they are of hyperbolic type.

If $\Delta=0$, then they are of linear type.

Thus, we obtained **the polar form** of 2×2 special matrix $S=\begin{pmatrix} a & b \\ c & d \end{pmatrix}=\begin{pmatrix} m+hb & b \\ kb & m-hb \end{pmatrix}$,

where $\det S=m^2-\Delta b^2=1$, using **argument** θ, commutative coefficient k and h as follows.

For $\Delta<0$, $S=S(\theta,k,h)=\begin{pmatrix} \cos\theta+\frac{h}{\sqrt{-\Delta}}\sin\theta & \frac{1}{\sqrt{-\Delta}}\sin\theta \\ \frac{k}{\sqrt{-\Delta}}\sin\theta & \cos\theta-\frac{h}{\sqrt{-\Delta}}\sin\theta \end{pmatrix}$	elliptic type	(14)
For $\Delta>0$, $S=S(\theta,k,h)=\begin{pmatrix} \cosh\theta+\frac{h}{\sqrt{\Delta}}\sinh\theta & \frac{1}{\sqrt{\Delta}}\sinh\theta \\ \frac{k}{\sqrt{\Delta}}\sinh\theta & \cosh\theta-\frac{h}{\sqrt{\Delta}}\sinh\theta \end{pmatrix}$	hyperbolic type	(15)
For $\Delta=0$, $S=S(b,h)=\begin{pmatrix} m+hb & b \\ -h^2b & m-hb \end{pmatrix}$, $m=\pm1$.	linear type	(16)

Any 2×2 non-diagonal regular matrix F is represented by the polar form of

$$F=(\det F)^{1/2}S(\theta,k,h) \quad \text{or} \quad F=(\det F)^{1/2}S(b,h). \tag{17}$$

But when $\det F<0$, the matrix F stands for a inside out transformation, then the orbit of invariant function as shown eq(13) branches off to a conjugate hyperbolic curve of $\phi\,(\boldsymbol{p})$ and complex number of argument θ comes out.

We obtained **the addition theorem** of argument θ from eq(14),(15),(16) as follows.

$$S(\theta_1,k,h)\,S(\theta_2,k,h)=S(\theta_1+\theta_2,k,h),\ \ S(\theta,k,h)^n=S(n\theta,k,h),\ \ S(\theta,k,h)^{-1}=S(-\theta,k,h).$$

$$S(b_1,h)\,S(b_2,h)=S(b_1+b_2,h),\ \ S(b,h)^n=S(nb,h),\ \ S(b,h)^{-1}=S(-b,h). \tag{18}$$

- **The norm** $\|\boldsymbol{p}\|$ of a vector $\boldsymbol{p}$ is defined by the invariant function $\phi\,(\boldsymbol{p})$ such that

$$\|\boldsymbol{p}\|^2=\phi\,(\boldsymbol{p})=-kx^2+y^2+2hxy, \text{ and the norm } \|\boldsymbol{p}\|=\phi\,(\boldsymbol{p})^{1/2}. \tag{20}$$

- **The inner product** of $\boldsymbol{p}$ and $\boldsymbol{q}$ is defined by the invariant function $\phi\,(\boldsymbol{p})$ as follows.

$$\boldsymbol{p}=(x_1,y_1),\ \ \boldsymbol{q}=(x_2,y_2)=F\boldsymbol{p}=(\det F)^{1/2}S(\theta,k,h)\boldsymbol{p}, \tag{21}$$

$$\|\boldsymbol{p}\|^2=\phi\,(\boldsymbol{p})=-kx_1^2+y_1^2+2hx_1y_1,\ \ \|\boldsymbol{q}\|^2=\phi\,(\boldsymbol{q})=-kx_2^2+y_2^2+2hx_2y_2,$$

$$\|\boldsymbol{p}+\boldsymbol{q}\|^2=\phi\,(\boldsymbol{p}+\boldsymbol{q})=-k(x_1+x_2)^2+(y_1+y_2)^2+2h(x_1+x_2)(y_1+y_2),$$

$$=\|\boldsymbol{p}\|^2+\|\boldsymbol{q}\|^2+2(\underline{-kx_1x_2+y_1y_2+h(x_1y_2+x_2y_1)}). \tag{22}$$

Thus, we obtain the inner product and deduce **the cosine theorems** as

$(\boldsymbol{p},\boldsymbol{q})=\boldsymbol{p}\cdot\boldsymbol{q}=-kx_1x_2+y_1y_2+h(x_1y_2+x_2y_1)=\boldsymbol{p}\cdot(\det F)^{1/2}S(\theta,k,h)\boldsymbol{p}$

$=(\|\boldsymbol{p}+\boldsymbol{q}\|^2-\|\boldsymbol{p}\|^2-\|\boldsymbol{q}\|^2)/2=(\phi(\boldsymbol{p}+\boldsymbol{q})-\phi(\boldsymbol{p})-\phi(\boldsymbol{q}))/2$ (23)

$=(\det F)^{1/2}\phi(\boldsymbol{p})\cos\theta=\|\boldsymbol{p}\|\ \|\boldsymbol{q}\|\cos\theta$ when S is elliptic type, (24)

$=(\det F)^{1/2}\phi(\boldsymbol{p})\cosh\theta=\|\boldsymbol{p}\|\ \|\boldsymbol{q}\|\cosh\theta$ when S is hyperbolic type. (25)

▪ Furthermore, when $d=a \Leftrightarrow h=0$ on a special linear transformation S, we define a commutative special **isodiagonal** transformation A and invariant function $\phi(\boldsymbol{p})$ as

$A=\begin{pmatrix} a & b \\ c & a \end{pmatrix}=\begin{pmatrix} a & b \\ kb & a \end{pmatrix}$, $\det A=a^2-kb^2=1$, $k=c/b$, where k is a commutative coefficient,

invariant function $\phi(A\boldsymbol{p})=\phi(\boldsymbol{p})=-kx^2+y^2$. (26)

In this case, we define the norm $\|\boldsymbol{p}\|$ and the inner product $(\boldsymbol{p},\boldsymbol{q})$ as follows.

$\|\boldsymbol{p}\|^2=\phi(\boldsymbol{p})=-kx^2+y^2$, $\|\boldsymbol{p}\|=\phi(\boldsymbol{p})^{1/2}$. (27)

$(\boldsymbol{p},\boldsymbol{q})=\boldsymbol{p}\cdot\boldsymbol{q}=-kx_1x_2+y_1y_2$. (28)

If $(\boldsymbol{p},\boldsymbol{q})=0 \Leftrightarrow (y_1/x_1)(y_2/x_2)=k$, then two vectors of $\boldsymbol{p}$ and $\boldsymbol{q}$ are orthogonal.

From matrix A with $\det A=1$, we obtained the polar form of A using argument θ,k.

When $k<0$, then $A=\begin{pmatrix} a & b \\ kb & a \end{pmatrix}=\begin{pmatrix} \cos\theta & \sin\theta/\sqrt{-k} \\ -\sqrt{-k}\sin\theta & \cos\theta \end{pmatrix}$, θ is elliptic angle. (30)

When $k=-1$, then this type of matrix A is called a rotation transformation.

When $k>0$, then $A=\begin{pmatrix} a & b \\ kb & a \end{pmatrix}=\begin{pmatrix} \cosh\theta & \sinh\theta/\sqrt{k} \\ \sqrt{k}\sinh\theta & \cosh\theta \end{pmatrix}$, θ is hyperbolic angle. (31)

This type of matrix A is called a Lorentz transformation.

When $k=0$, then $A=\begin{pmatrix} a & b \\ kb & a \end{pmatrix}=\begin{pmatrix} a & b \\ 0 & a \end{pmatrix}$, $a=\pm1$. (32)

This type of matrix A is called a Gallilean transformation.

2. Geometric Structure of a Line

Put two number lines 1,2 on one line, and make them coincide with their origins. The relation between their x-coordinates of x_1 and x_2 is

$x_2=rx_1 \Leftrightarrow x_1=r^{-1}x_2$, where r is a proportional constant.

To be equivalent of two number lines with x_1 and x_2-coordinates, we must have

$r=r^{-1} \Leftrightarrow r^2=1 \Leftrightarrow r=\pm1$. (33)

▪ When $r=-1$, then the two number lines are inverse of each other, and this type of a line is isotropic. A space line fits into this category.

▪ When $r=1$, then the two number lines are coincident with each other, and this type of a line is one way. A time line fits into this category.

▪ When $r\neq\pm1$, then the two number lines are similar.

3. Geometric Structure of a Plane

Theorem : A back and front symmetric plane is a linear space.

Brief proof【 From the definition of the inverse invariance of a line, we have that a line is a linear space. Also from the definition of a plane, we obtained at least two lines exist in a plane. Then, from the inverse invariance of a plane, we observed that these lines make a plane linear. □】

Put oblique right-hand coordinate systems on both face sides of a plane, and make them coincide with their origins. We define 2×2 rear surface coordinate transformation matrix B as inside out transformation, then det $B < 0$.

Any point $\boldsymbol{p}_i=(x_i,y_i)$, i=1,2,3,...on the front side corresponds to the point $\boldsymbol{q}_i=(u_i,v_i)$ on the back side as $\boldsymbol{q}_i= B\boldsymbol{p}_i$. We also have the point $\boldsymbol{q}_i= B\boldsymbol{p}_i$ on the front side which corresponds to the point $\boldsymbol{r}_i$ on the back side as $\boldsymbol{r}_i=B\boldsymbol{q}_i= B^2\boldsymbol{p}_i$. If $\boldsymbol{r}_i=\boldsymbol{p}_i$, then the back and front $\boldsymbol{p}_i$s are equivalent as vice varsa and it is indistinguishable which side of this plane is back or front. If the point $\boldsymbol{r}_i\neq\boldsymbol{p}_i$, then this plane is asymmetric. Therefore, the symmetric plane equation is

$$\boldsymbol{r}_i= B^2\boldsymbol{p}_i=\boldsymbol{p}_i \quad\Leftrightarrow\quad B^2= E \quad\Leftrightarrow\quad B = B^{-1}, \quad \text{where } \det B < 0. \tag{34}$$

We obtained an oblique reflection matrix B with 2 degree of freedom.

$$B=\pm\begin{pmatrix} a & -b \\ c & -a \end{pmatrix},\ \det B = -a^2+bc=-1,\ \text{eigen values } \lambda=\pm 1. \tag{35}$$

The matrix B has the following properties. We shall treat negative solution $-B$ later.

- The matrix B has an eigen value $\lambda=1$, then it has an invariant line f($\boldsymbol{p}$) given as eq(2).

$$\mathrm{f}(B\boldsymbol{p})=\mathrm{f}(\boldsymbol{p})=cx-(a-1)y. \tag{2}$$

- For $\lambda=1 \Leftrightarrow B\boldsymbol{p}=\boldsymbol{p}$, this line is called a fold line f: $\quad cx-(a+1)y=0. \quad (36)$
- For $\lambda=-1 \Leftrightarrow B\boldsymbol{p}=-\boldsymbol{p}$, this line is called an isotropic line g : $cx-(a-1)y=0. \quad (37)$

This isotropic line g is parallel to an invariant line f($\boldsymbol{p}$) shown as eq(2).

When a point $\boldsymbol{p}$ is in an invariant line f($\boldsymbol{p}$), and the point $\boldsymbol{r}$ is the intersection point of a fold line f and an invariant line f($\boldsymbol{p}$), then in the fold line f, $B\boldsymbol{r}=\boldsymbol{r}$, and in the invariant line, f($B\boldsymbol{p}$)=f($\boldsymbol{p}$)=f($\boldsymbol{r}$). Obtained by translating the vector $(\boldsymbol{p}-\boldsymbol{r})$ onto the isotropic line g,

$$B(\boldsymbol{p}-\boldsymbol{r})=-(\boldsymbol{p}-\boldsymbol{r}) \quad\Leftrightarrow\quad B\boldsymbol{p}-\boldsymbol{r}=-\boldsymbol{p}+\boldsymbol{r} \quad\Leftrightarrow\quad B\boldsymbol{p}+\boldsymbol{p}=2\boldsymbol{r}. \tag{38}$$

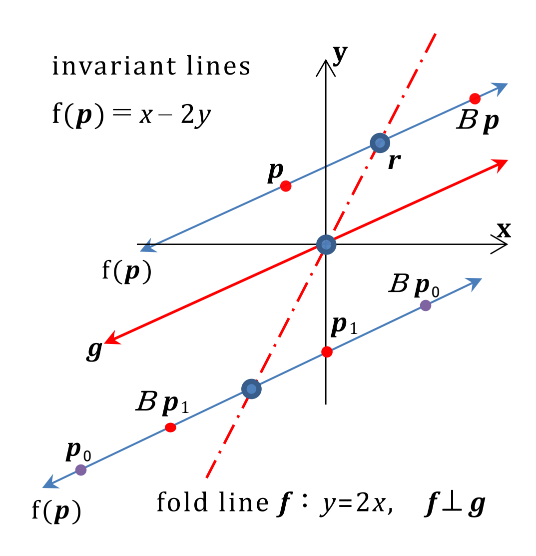
Since the fixed-point $\boldsymbol{r}$ is the middle point of the point $\boldsymbol{p}$ and $B\boldsymbol{p}$, and each invariant line f($\boldsymbol{p}$) is parallel to the isotropic line g, then the invariant lines f($\boldsymbol{p}$) are isotropic. This plane with back and front indistinguishable surface has innumerable oblique reflection planes centered around any one point on itself.

An oblique reflection plane is an eigen plane with eigen line of f and g. These two lines are orthogonal, but commonly seem not perpendicular.

Figure1 shows the case of $B=\frac{1}{3}\begin{pmatrix} -5 & 4 \\ -4 & 5 \end{pmatrix}$.

- Meanwhile, when $B\neq B^{-1}$, then we will have another geometry on an asymmetric plane.

Figure1 Oblique Reflection Plane



▪ We derive a special **isodiagonal** transformation A from the oblique reflection transformation B and the reflection matrix M.

$$A=BM=\begin{pmatrix} a & -b \\ c & -a \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}=\begin{pmatrix} a & b \\ c & a \end{pmatrix}=\begin{pmatrix} a & b \\ kb & a \end{pmatrix}, \text{ where } M=\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \det A=1,\ k=c/b. \quad (40)$$

The matrix A is a coordinate transformation between the right-hand systems in which the y-coordinate on the rear side is reflected in x-axis by the matrix M. (→see Figure2)

Since $B=AM$, $\phi(M\boldsymbol{p})=\phi(\boldsymbol{p})$, and $\phi(A\boldsymbol{p})=\phi(\boldsymbol{p})=-kx^2+y^2$ from eq(26), then the oblique reflection matrix B has the same invariant function $\phi(\boldsymbol{p})$ of A as given by

$$\phi(B\boldsymbol{p})=\phi(A(M\boldsymbol{p}))=\phi(M\boldsymbol{p})=\phi(\boldsymbol{p})=\phi(A\boldsymbol{p})=-kx^2+y^2. \quad (41)$$

When commutative coefficient k is fixed, then we observed that any combination of the Bs and As matrices has a common invariant function $\phi(\boldsymbol{p})$, and their join operation is closed in the orbit of $\phi(\boldsymbol{p})$ given below.

$$\phi(BA^2\cdots B^{-1}A^{-1}\boldsymbol{p})=\phi(A^2\cdots B^{-1}A^{-1}\boldsymbol{p})=\phi(A\cdots B^{-1}A^{-1}\boldsymbol{p})=\phi(B^{-1}A^{-1}\boldsymbol{p})$$
$$=\phi(BA^{-1}\boldsymbol{p})=\phi(A^{-1}\boldsymbol{p})=\phi(\boldsymbol{p})=\phi(E\boldsymbol{p})=-kx^2+y^2. \quad (42)$$

Thus, we conclude that the Bs and As matrices with any matrix of these combination makes **isometric transformation group** on the orbit of invariant function $\phi(\boldsymbol{p})$ with the isometric $\|B\boldsymbol{p}\|^2=\|A\boldsymbol{p}\|^2=\|\boldsymbol{p}\|^2=\phi(B\boldsymbol{p})=\phi(A\boldsymbol{p})=\phi(\boldsymbol{p})=-kx^2+y^2$.

The oblique reflection matrix B transforms a point $\boldsymbol{p}$ on the front surface to corresponding rear point $\boldsymbol{q}$ on the back surface as

$$\boldsymbol{q}_1=B\boldsymbol{p}_1,\quad \boldsymbol{q}_2=B\boldsymbol{p}_2. \quad (44)$$

However, on the front surface, we obtained a figure transformation matrix X transforms a point from $\boldsymbol{p}_1$ to $\boldsymbol{p}_2$. Thus, $\boldsymbol{p}_2=X\boldsymbol{p}_1$, $\det X>0$. (45)

Also on the back surface, we obtained that the matrix Y transforms a point from $\boldsymbol{q}_1$ to $\boldsymbol{q}_2$. Thus, $\boldsymbol{q}_2=Y\boldsymbol{q}_1$, $\det Y>0$. (46)

Consequently from these four equations, we obtained the relation :

$$\boldsymbol{q}_2=Y\boldsymbol{q}_1=\underline{YB}\boldsymbol{p}_1=B\boldsymbol{p}_2=\underline{BX}\boldsymbol{p}_1. \quad (47)$$

Meanwhile, since the point $\boldsymbol{p}_1$ is arbitrary, and $B=B^{-1}$, then we obtained the relation :

$$YB=BX \iff Y=BXB \iff BY=XB,\ \det Y=\det X>0,\ \operatorname{tr} Y=\operatorname{tr} X. \quad (48)$$

Therefore the matrix Y is similar to X, and they are the same type of matrix. Then, using $B=M$ (M is one of the solutions of $B=B^{-1}$) and $B=AM$ in eq(48), we obtained

$$YM=MX,\qquad YAM=A\underline{MX}=A\underline{YM}. \quad (50)$$

Comparing the first and third side, we had that $YA=AY$. In the same way $XA=AX$.

However, since the matrix A and the figure transformation matrix X, Y are commutative, then the matrix X and Y have a common relative invariant function as

$$\phi(X\boldsymbol{p})=\det X\,\phi(\boldsymbol{p})=\phi(Y\boldsymbol{p})=\det Y\,\phi(\boldsymbol{p}),\text{ where } \phi(A\boldsymbol{p})=\phi(\boldsymbol{p})=-kx^2+y^2. \quad (51)$$

Thus we conclude that the matrix As and Xs and Ys with any matrix of these combination make commutative transformation group on both sides of a plane.

Furthermore, the matrix Bs and As and Xs and Ys with any matrix of these combination make transformation group based on the orbit of invariant function $\phi(\boldsymbol{p})$ on both sides of a plane. Euclidean geometry is one of a sub group of this group.

On the other hand, based on the sign of k, we obtained that the existing direction of fold lines and isotropic lines vary on the coordinate system. Thus, we have the following cases.
(1) If $k < 0$, then matrix A is an elliptic type, and from eq(30), we express the matrix $B=AM$ as follows:

$$B=\begin{pmatrix} a & -b \\ c & -a \end{pmatrix}=\begin{pmatrix} \cos\theta & -\sin\theta/\sqrt{-k} \\ -\sqrt{-k}\sin\theta & -\cos\theta \end{pmatrix},\quad \det B=-1,\ b=\frac{1}{\sqrt{-k}}\sin\theta,\ c=kb,\ \theta\in \text{Real}. \quad (52)$$

From eq(36), we have the fold line f: $y=\frac{c}{a+1}x=-\sqrt{-k}\frac{\sin\theta}{\cos\theta+1}x=-\sqrt{-k}\tan\frac{\theta}{2}\cdot x=ux$.(53)

From eq(37), we have the isotropic line g : $y=\frac{c}{a-1}x=-\sqrt{-k}\frac{\sin\theta}{\cos\theta-1}x=\sqrt{-k}\cot\frac{\theta}{2}\cdot x=vx.$ (54)

The existing direction of lines f and g : $-\infty < u < \infty,\ -\infty < v < \infty,\ uv=k.$ (55)

We observed that an oblique reflection plane made of a fold line f and an isotropic line g exists in all directions on this symmetric plane. This is similar to the case of a negative solution of the matrix $-B$. Therefore, we obtained that this symmetric plane which is made of oblique reflection planes is completely isotropic. This type of a symmetric plane which has the oblique reflection matrix B with $k < 0$ fits in space*space plane and forms an elliptic type plane geometry.
When $k=-1$, we call this type of plane an Euclidean plane.

(2) If $k > 0$, then matrix A is a hyperbolic type, and from eq(31), we express the matrix B as follows:

$$B=\begin{pmatrix} a & -b \\ c & -a \end{pmatrix}=\begin{pmatrix} \cosh\theta & -\sinh\theta/\sqrt{k} \\ \sqrt{k}\sinh\theta & -\cosh\theta \end{pmatrix},\quad \det B=-1,\ b=\frac{1}{\sqrt{k}}\sinh\theta,\ c=kb,\ \theta\in \text{Real}. \quad (56)$$

From eq(36), we have the fold line f : $y=\frac{c}{a+1}x=\sqrt{k}\frac{\sinh\theta}{\cosh\theta+1}x=\sqrt{k}\tanh\frac{\theta}{2}\cdot x=u\text{x}.$ (57)

From eq(37), we have the isotropic line g : $y=\frac{c}{a-1}x=\sqrt{k}\frac{\sinh\theta}{\cosh\theta-1}x=\sqrt{k}\coth\frac{\theta}{2}\cdot x=v\text{x}.$ (58)

The asymptote line : $y=\pm\sqrt{k}\text{x},\quad \theta\to\pm\infty.$ (60)

The existing direction of the lines f: $-\sqrt{k} < u < \sqrt{k}$. right and left quadrant. (61)

The existing direction of the lines g : $-\infty < v < -\sqrt{k}$ and $\sqrt{k} < v < \infty$. upper and lower quadrant. Note : $uv=k$, f and g is perpendicular. (62)

The direction of fold lines f exists on the right and left quadrant regions, and the direction of isotropic lines g exists on the upper and lower quadrant regions on the coordinate system. The inverse relation of g and f is the case of the negative solution of the matrix $-B$. Therefore, we obtained that this symmetric plane which is made of oblique reflection planes is semi isotropic, as the time axis is a fold line f, and the space axis is an isotropic line g. This type of a symmetric plane which has the oblique reflection matrix B with $k > 0$ fits in space*time plane, and forms a hyperbolic type plane geometry.
However, when $k=1/c^2$ and $y=t$, we call this hyperbolic type plane geometry a Minkowskian spacetime geometry[6]. The constant c represents the speed of light, and variables x and t are space and time respectively.

4. Expected Conclusions

Since a point in a plane is not only existent as a linear coordinate point but also holding the symmetry condition of a plane from where **the metrical structure** of a plane inevitably comes out. The back and front symmetric plane is made of the oblique reflection planes in which there are space*space type plane or space*time type plane.

Thus, we draw the following conclusions.

- The Pythagorean theorem depends on the symmetry of a space*space plane, such that the squared norm of a vector $\boldsymbol{p}$ is invariant as $\|\boldsymbol{p}\|^2=\phi(\boldsymbol{p})=-kx^2+y^2=x^2+y^2$, where $k=-1$.
- The special principle of relativity is based on the symmetry of a space*time type plane, because the Lorentz transformation comes out from this plane, and any basic law of nature with position vectors must be subject to this symmetry.

Eq(60) shows the existence of the universal constant of $x/y=\pm 1/\sqrt{k}$, where $1/\sqrt{k}$ represents the maximum space/time ratio in the universe.

- The arrow of time problem in physics would be explained since a space*time plane in the cosmos has the symmetry of the back and front surface.

5. An Example of Euclidean Plane

When $k=-1$, then we obtained that the invariant function is a circle $\phi(A\boldsymbol{p})=\phi(\boldsymbol{p})=x^2+y^2$, and this is similar to the case of the Euclidean geometry[1]. Meanwhile, we obtained the matrix A is a rotation transformation matrix, and the matrix B is a reflection transformation matrix. In this case, a fold line f and an isotropic line g are perpendicular. However, from eq(20), when rotation angle is $\theta=-\pi/3$, then

$$B=\frac{1}{2}\begin{pmatrix}1 & \sqrt{3}\\ \sqrt{3} & -1\end{pmatrix},\quad A=BM=\begin{pmatrix}a & b\\ c & a\end{pmatrix}=\begin{pmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{pmatrix}=\frac{1}{2}\begin{pmatrix}1 & -\sqrt{3}\\ \sqrt{3} & 1\end{pmatrix},$$

$$\text{where } M=\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix},\quad c=-b=-\sin\theta=\sqrt{3}/2,\quad a=\cos\theta=1/2. \tag{63}$$

Fold line f: $\quad y=-\tan\frac{\theta}{2}\cdot x=\tan\frac{\pi}{6}\cdot x=\frac{x}{\sqrt{3}}$. (64)

Isotropic line g: $\quad y=\cot\frac{\theta}{2}\cdot x=-\cot\frac{\pi}{6}\cdot x=-\sqrt{3}x.$ (65)

Back surface x_2-axis : $\quad y=\frac{c}{a}x=-\tan\theta\cdot x=\tan\frac{\pi}{3}\cdot x=\sqrt{3}x,\qquad y_2$-axis : $y=-\frac{x}{\sqrt{3}}$. (66)

Considering the front surface, we have that the coordinate transformation matrix A has two coordinate axes namely, $\mathbf{x}_A$-axis : $y=-\sqrt{3}x$, and $\mathbf{y}_A$-axis : $y=x/\sqrt{3}$. (67)

When X is a figure transformation matrix on the front surface, consider any point $\boldsymbol{p}$ is transformed such that combinations of the B, A, X matrices make a closed circuit.

For $X=\det X\cdot S$, $\det S=1$, in the invariant function $\phi(A\boldsymbol{p})=\phi(B\boldsymbol{p})=\phi(S\boldsymbol{p})=\phi(\boldsymbol{p})=x^2+y^2$.

We showed the coordinate transformation matrix A as if the figure tansformation matrix A^{-1} in order to understand easily the result of transformation of point $\boldsymbol{p}$.

- $\boldsymbol{p}$(front surface)→ $A\boldsymbol{p}$(front)→ $BA\boldsymbol{p}$(back)→ $ABA\boldsymbol{p}=B\boldsymbol{p}$(back)→ $BABA\boldsymbol{p}=\boldsymbol{p}$(front)
- $\boldsymbol{p}$(front)→ $S\boldsymbol{p}$(front)→ $BS\boldsymbol{p}$(back)→ $SBS\boldsymbol{p}=B\boldsymbol{p}$(back)→ $BSBS\boldsymbol{p}=\boldsymbol{p}$(front)

▪ $\boldsymbol{p}$(front)→ $X\boldsymbol{p}$ = det X・ $S\boldsymbol{p}$(front)→ $B\ X$ = det X・$BS\boldsymbol{p}$(back)

→ $SBS\boldsymbol{p}$ = $B\boldsymbol{p}$(back)→ $BSBS\boldsymbol{p}$ = $\boldsymbol{p}$(front) (68)

In the figure2, $0 < \det X < 1$ is proposed. On the back side, we obtained that the point $B\boldsymbol{p}$ is hidden behind $\boldsymbol{p}^{\{1\}}$ on the front side, and similarly, $\boldsymbol{p}$ is hidden behind $B\boldsymbol{p}^{\{2\}}$. The two points of $\boldsymbol{p}$s have the same coordinate value, but their coordinate system is different, x-y axis on the front side, and x_2-y_2 axis on the back side.

We also obtained that the fold line f, the isotropic line g, the invariant lines f($\boldsymbol{p}$), and the invariant function ϕ ($\boldsymbol{p}$), have the same shape from each side of the coordinate system.

The solid line ($\boldsymbol{p}$→ $A\boldsymbol{p}$ or $\boldsymbol{p}$→ $X\boldsymbol{p}$) on the front side is an equivalent transformation to the broken line ($B\boldsymbol{p}$→ $BA\boldsymbol{p}$ or $B\boldsymbol{p}$→ $BX\boldsymbol{p}$) on the back side respectively.

Both sides of the Euclidean plane are symmetric when compared by viewing the surface from each back and front side.

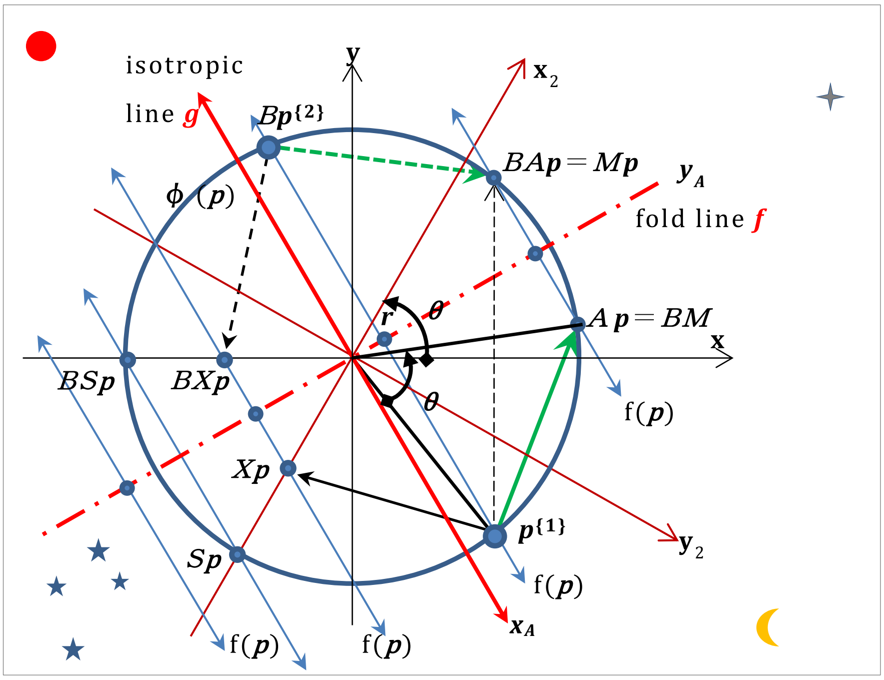


Figure2 Symmetry Transformation on a Euclidean Plane

References

[1]David Hilbert "Grundlagen der Geometrie", 1899

[2]Tsuruichi Hayashi,"Form of Elementary Geometry", 1911

→ http://fomalhautpsa.sakura.ne.jp/Science/Other/kikagaku-teisai.pdf

Jun Tosaka,"Geometry and Space", 1926

→ https://www.aozora.gr.jp/cards/000281/files/43263_35546.html

[3]Henri Poincaré,"La hyposesis Science ",1902

and "La Valuer de la Science",1905 Chap3.1 Concept of relative space

It is indistinguishable when one world changes the coordinate axis or the scale of length

into another world. ・・・・・・

Poincaré was persuiting the geometrical theory to the results of M-M experiment.

[4]Albert Einstein, "On the electorodynamics of moving bodies" *Ann. Phys 17,* 1905

[5]Hiroaki Fujimori,"Mathematical Principle of the Special Theory of Relativity",2012,
private book ; → http://iss.ndl.go.jp/books/R100000002-I023840492-00
or web site → http://spatim.sakura.ne.jp/english.html

[6]Herman Minkowski, "Raum und Zeit", *Phys. Zeitschr.10*, 104, 1909

Appendix A : An Example of Geometry on Symmetry Plane

Nine-point circle theorem on both Minkowski plane and Euclidean plane.

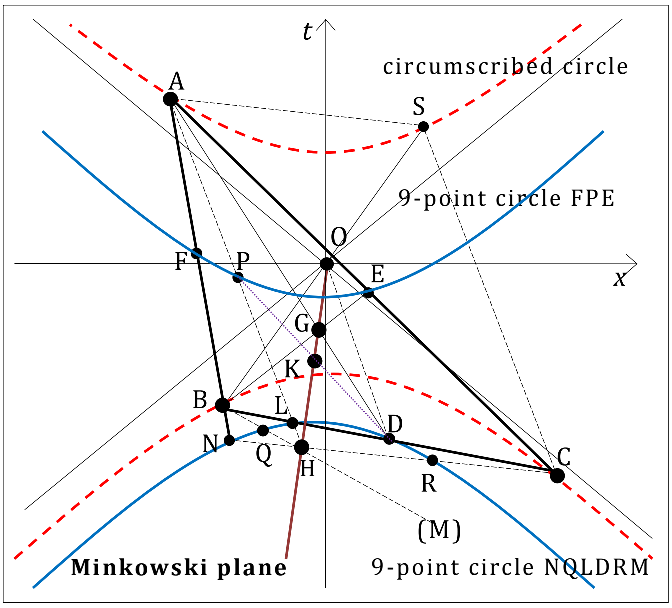
Point O is the center of circumscribed circle on △ABC.

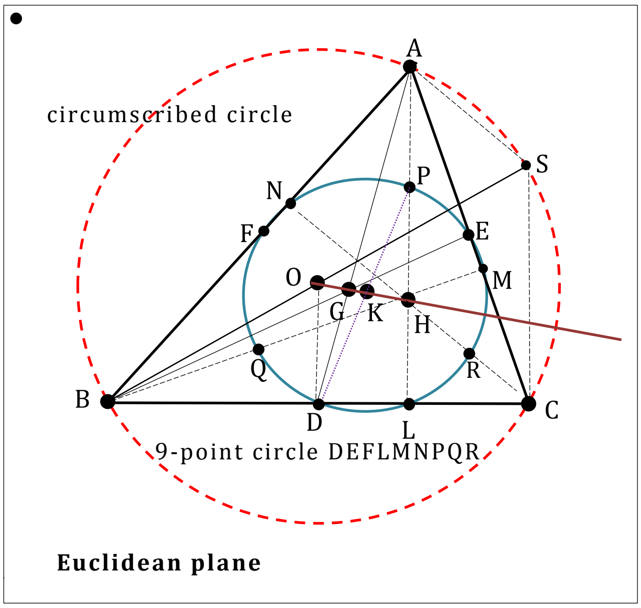
Point G is the center of gravity of △ABC.

Point H is the crossing point of perpendicular lines of AL,BM,CN.

Point K is the middle point of O and H, and the center of 9-point circle.

Line OH is the Euler line.



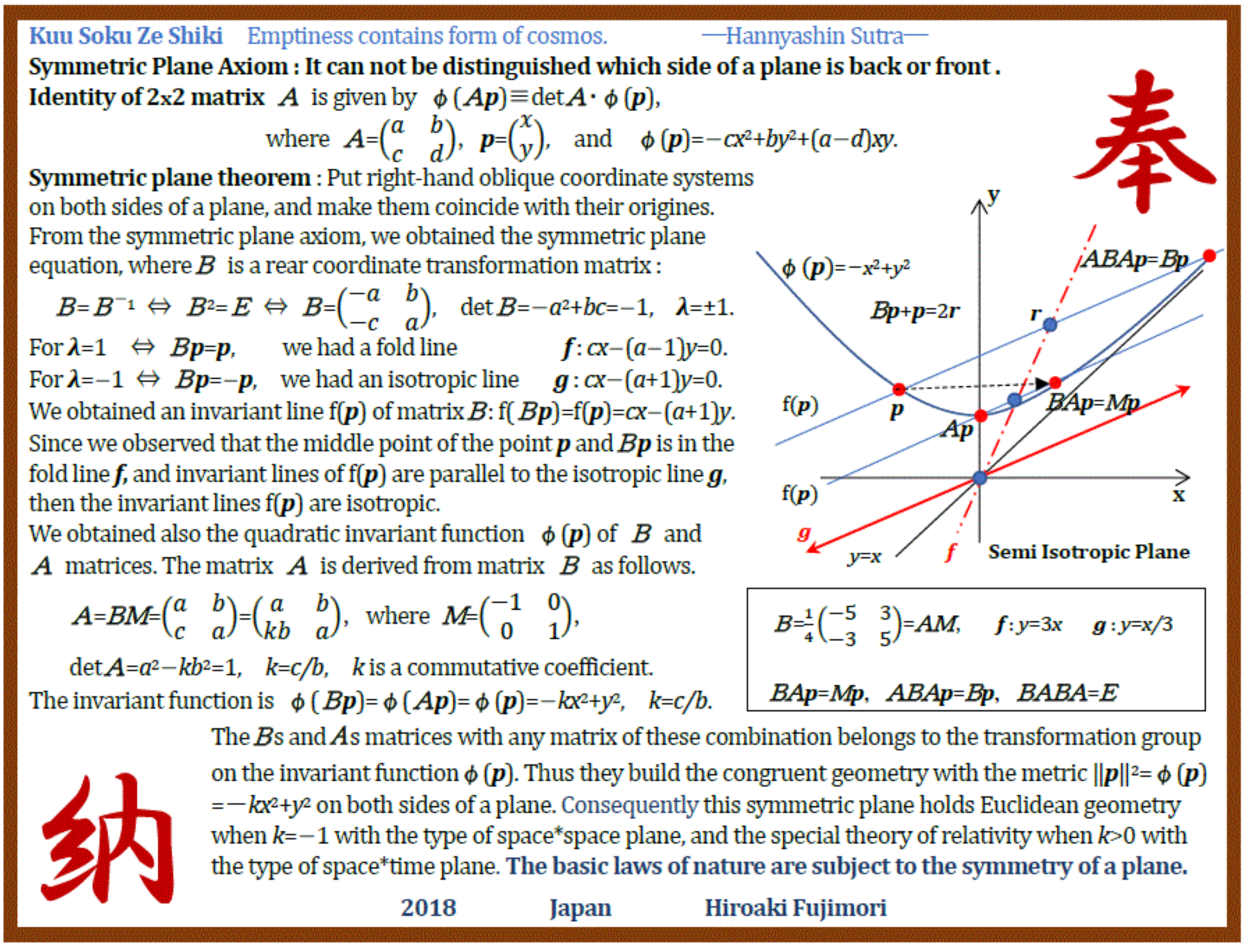


Symmetry plane geometry is constructed and deduced uniformly on both Euclidean plane and Minkowski plane.

→ http://spatim.sakura.ne.jp/4syo.pdf last page

Appendix B : Sangaku Presenred on Website

"**Sangaku**" is a tradition of Japan from the Edo era. When new problem of geometry was solved or found, the wooden framed solution with figures was put on board at the shrine or the temple. 奉納"Houno" means dedication and presentation.



Kuu Soku Ze Shiki　Emptiness contains form of cosmos.　—Hannyashin Sutra—

Symmetric Plane Axiom : It can not be distinguished which side of a plane is back or front .

Identity of 2x2 matrix A is given by $\phi(Ap)\equiv\det A\cdot\phi(p)$,

where $A=\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $p=\begin{pmatrix} x \\ y \end{pmatrix}$, and $\phi(p)=-cx^2+by^2+(a-d)xy$.

Symmetric plane theorem : Put right-hand oblique coordinate systems on both sides of a plane, and make them coincide with their origines. From the symmetric plane axiom, we obtained the symmetric plane equation, where B is a rear coordinate transformation matrix :

$$B=B^{-1}\Leftrightarrow B^2=E\Leftrightarrow B=\begin{pmatrix} -a & b \\ -c & a \end{pmatrix},\quad \det B=-a^2+bc=-1,\quad \lambda=\pm1.$$

For $\lambda=1$ $\Leftrightarrow$ $Bp=p$, we had a fold line $f: cx-(a-1)y=0$.

For $\lambda=-1$ $\Leftrightarrow$ $Bp=-p$, we had an isotropic line $g: cx-(a+1)y=0$.

We obtained an invariant line f(p) of matrix B: f(Bp)=f(p)=$cx-(a+1)y$.

Since we observed that the middle point of the point p and Bp is in the fold line f, and invariant lines of f(p) are parallel to the isotropic line g, then the invariant lines f(p) are isotropic.

We obtained also the quadratic invariant function $\phi(p)$ of B and A matrices. The matrix A is derived from matrix B as follows.

$$A=BM=\begin{pmatrix} a & b \\ c & a \end{pmatrix}=\begin{pmatrix} a & b \\ kb & a \end{pmatrix},\quad \text{where } M=\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix},$$

$\det A=a^2-kb^2=1$, $k=c/b$, k is a commutative coefficient.

The invariant function is $\phi(Bp)=\phi(Ap)=\phi(p)=-kx^2+y^2$, $k=c/b$.

$B=\frac{1}{4}\begin{pmatrix} -5 & 3 \\ -3 & 5 \end{pmatrix}=AM$, $f: y=3x$　$g: y=x/3$

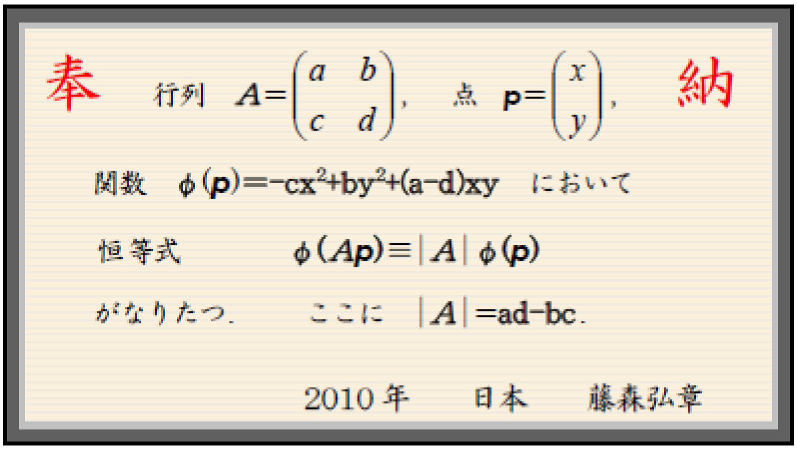
$BAp=Mp$, $ABAp=Bp$, $BABA=E$

The Bs and As matrices with any matrix of these combination belongs to the transformation group on the invariant function $\phi(p)$. Thus they build the congruent geometry with the metric $\|p\|^2=\phi(p)$ $=-kx^2+y^2$ on both sides of a plane. Consequently this symmetric plane holds Euclidean geometry when $k=-1$ with the type of space*space plane, and the special theory of relativity when $k>0$ with the type of space*time plane. **The basic laws of nature are subject to the symmetry of a plane.**

2018　Japan　Hiroaki Fujimori

http://www.spatim.sakura.ne.jp/　Although nature seems strange, the truth is trivial.　sangaku.gif

Identity of 2x2 matrix



奉　行列 $A=\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, 点 $p=\begin{pmatrix} x \\ y \end{pmatrix}$, 納

関数 $\phi(p)=-cx^2+by^2+(a-d)xy$ において

恒等式 $\phi(Ap)\equiv|A|\,\phi(p)$

がなりたつ. ここに $|A|=ad-bc$.

2010 年　日本　藤森弘章